Solutions for Econometrics I Homework No.3

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Exercise 3.1

We have the following model:

$$y_{TN\times 1} = X_{TN\times Nk}\beta_{Nk\times 1} + u_{TN\times 1}$$

In Matrix notation this looks like:

$$\begin{pmatrix} y_{11} \\ \vdots \\ y_{1T} \\ \vdots \\ \vdots \\ y_{N1} \\ \vdots \\ y_{NT} \end{pmatrix} = \begin{pmatrix} X_{1_{T\times k}} 0 & \dots & \dots & \dots & \dots & & 0 \\ 0 & X_{2_{T\times k}} & & & & & \vdots \\ \vdots & \dots & \ddots & & & & \vdots \\ \vdots & \dots & \dots & \ddots & & & \vdots \\ \vdots & \dots & \dots & \dots & \ddots & & \vdots \\ \vdots & & & & & \ddots & 0 \\ 0 & \dots & & 0 & \dots & \dots & X_{N_{T\times k}} \end{pmatrix} \begin{pmatrix} \beta_{1_{k\times 1}} \\ \beta_{2_{k\times 1}} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \beta_{N_{k\times 1}} \end{pmatrix} + \begin{pmatrix} u_{11} \\ \vdots \\ u_{1T} \\ \vdots \\ \vdots \\ u_{N1} \\ \vdots \\ u_{NT} \end{pmatrix}$$

The cont. correlation among the observations $\mathbf{E}(u_{i,t}u_{j,t}) = \sigma_{ij}$ and $\mathbf{E}(u_{it}^2) = \sigma_i^2$ results into a VCV of the following form:

$$\Sigma_{uu_{TN\times TN}} = \begin{pmatrix} \sigma_1^2 I_T & \sigma_{1,2} I_T & \dots & \dots & \sigma_{1N} I_T \\ \vdots & \sigma_2^2 I_T & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \\ \sigma_{N,1} I_T & \dots & \dots & & \sigma_N^2 I_T \end{pmatrix}$$

To write the VCV in more compact form we introduce $\Sigma_{0_{N\times N}}$ given by:

$$\Sigma_{0_{N\times N}} = \begin{pmatrix} \sigma_1^2 & \sigma_{1,2} & \dots & \dots & \sigma_{1N} \\ \vdots & \sigma_2^2 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \\ \sigma_{N,1} & \dots & \dots & & \sigma_N^2 \end{pmatrix}$$

We can write now $\Sigma_{uu_{TN\times TN}} = (\Sigma_{0N\times N} \otimes I_T)$.

The generalized least squares estimator is given by:

$$\tilde{\beta}_{GLS} = [X'(\Sigma_0^{-1} \otimes I_T)X]^{-1}[X'(\Sigma_0^{-1} \otimes I_T)y]$$

Show that if $X_1 = X_2 = X_3 = \dots = X_N = X$ then $\tilde{\beta}_{GLS}$ coincides with the OlS estimator (for N seperate OLS Regressions).

The X-matrix ($X_{TN\times Nk}$)looks now like

$$\begin{pmatrix} X & 0 & \dots & \dots & 0 \\ 0 & X & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \dots & & 0 & X \end{pmatrix}$$

which can be rewritten as $X = (I_N \otimes X_{T\times k})$. The GLS estimator then becomes:

$$\tilde{\beta}_{GLS} = [X'(\Sigma_0^{-1} \otimes I_T)X]^{-1}[X'(\Sigma_0^{-1} \otimes I_T)y] \tag{1}$$

$$= [(I_N \otimes X'_{k\times T})(\Sigma_{0_{N\times N}}^{-1} \otimes I_T)(I_N \otimes X_{T\times k})]^{-1}[(I_N \otimes X')(\Sigma_0^{-1} \otimes I_T)y_{N\times 1}] \tag{2}$$

$$= [(\Sigma_{0_{N\times N}}^{-1} \otimes X'_{k\times T})(I_N \otimes X_{T\times k})]^{-1}[(\Sigma_0^{-1} \otimes X')y] \tag{3}$$

$$= [(\Sigma_{0_{N\times N}}^{-1} \otimes X'X_{k\times k})]^{-1}[(\Sigma_0^{-1} \otimes X')y] \tag{4}$$

$$= [\Sigma_{0_{N\times N}}^{-1} \otimes (X'X_{k\times k})^{-1}][(\Sigma_0^{-1} \otimes X')y] \tag{5}$$

$$= [\underbrace{\Sigma_{0_{N\times N}}\Sigma_{0_{N\times N}}^{-1}}_{I_N} \otimes (X'X)^{-1}X']y \tag{6}$$

$$= (I_N \otimes (X'X)^{-1}X')y \tag{7}$$

$$= \begin{pmatrix} (X'X)^{-1}X'y_1 \\ \vdots \\ \vdots \\ (X'X)^{-1}X'y_N \end{pmatrix} \tag{8}$$

Where we have used the following rules concerning the Kronecker product: For A, B nonsingular it holds that

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

For A, B with matching dimensions

$$(A \otimes B)(C \otimes D) = (AC \otimes BD).$$

Exercise 3.2

(i) *Show that the function* $f(y) = \frac{1}{\sqrt{2\pi}} exp\left(-\frac{y^2}{2}\right)$ *is a density function, i.e. show that it integrates to one over* $\mathbb{R}$.

Show that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy = \sqrt{2\pi}$$

One may split up the integral on the left-hand side:

$$\int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy = \int_{-\infty}^{0} \exp\left(-\frac{y^2}{2}\right) dy + \int_{0}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy = 2 \int_{0}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy \quad (9)$$

Now take the square of it and rename y into t in one factor. Since the integral with respect to t can be considered constant with respect to y, it may by put into the integral with respect to y:

$$\begin{aligned} \left(\int_{0}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy\right)^2 &= \left(\int_{0}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt\right) \left(\int_{0}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy\right) = \\ &= \int_{0}^{\infty} \left(\int_{0}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt\right) \exp\left(-\frac{y^2}{2}\right) dy \end{aligned} \quad (10)$$

Now substitute in (10) for $t = xy$, which implies "$dt = y\ dx$". (Note: we substitute for t since it is in the "inner integral" and thus y can be regarded as constant (? right?).)

$$\int_{0}^{\infty} \left(\int_{0}^{\infty} y \exp\left(-\frac{x^2y^2}{2}\right) dx\right) \exp\left(-\frac{y^2}{2}\right) dy = \int_{0}^{\infty}\int_{0}^{\infty} \exp\left(-\frac{(1+x^2)y^2}{2}\right) y\ dx\ dy$$

By Fubini's Theorem*, we may interchange the $\int$-signs in (10).

$$= \int_{0}^{\infty}\int_{0}^{\infty} y \exp\left(-\frac{(1+x^2)y^2}{2}\right) dy\ dx$$

Now set $z := \frac{\sqrt{1+x^2}y}{\sqrt{2}}$, i.e $y = z\sqrt{\frac{2}{1+x^2}}$ and $dy = \sqrt{\frac{2}{1+x^2}}dz$:

$$= 2 \int_{0}^{\infty} \frac{1}{1+x^2} \int_{0}^{\infty} z \exp\left(-z^2\right) dz\ dx$$

We know the antiderivative of the "inner integral":

$$\int_{0}^{\infty} z \exp\left(-z^2\right) dz = \left|-\exp\left(-z^2\right)\right|_0^\infty = \frac{1}{2}$$

Substituting into the entire expression, and "remembering" that the resulting formula is the definition of arc tan yields the following expression for (10):

$$= \int_0^\infty \frac{1}{1+x^2} dx = |\text{arc tan } x|_0^\infty = \frac{\pi}{2}$$

So we know that

$$\left(\int_0^\infty \exp\left(-\frac{y^2}{2}\right) dy \right)^2 = \frac{\pi}{2}$$

Since the expression $\exp\left(-\frac{y^2}{2}\right)$ is non-negative over the interval $[0, \infty]$, the whole expression (9) (and its integral) is non-negative, therefore the root of it is non-negative as well

$$\int_{-\infty}^\infty \exp\left(-\frac{y^2}{2}\right) dy = 2 \int_0^\infty \exp\left(-\frac{y^2}{2}\right) dy = 2\sqrt{\frac{\pi}{2}} = \sqrt{2\pi}$$

which is what we aimed to show.

Any non-negative Lebesgue-integrable function $f(x)$ on $\mathbb{R}$ with total integral over $\int_{\mathbb{R}} f(x)dx$ is a density function for some probability distribution and vice versa.

Note: we could equally have used Fubini's Theorem in order to transform $(\int_{-\infty}^{\infty} \exp(-\frac{y^2}{2})dy)^2$ into polar coordinates (see `http://en.wikipedia.org/wiki/Gaussian_integral`).

* *Outline of use of Fubini's Theorem in this exercise:* Let μ and ν some σ-finite measures on $(\Omega_1, \mathcal{B}_1)$ and $(\Omega_2, \mathcal{B}_2)$ respectively. Further let $f(x,y) \in \mathcal{L}^+((\Omega_1 \otimes \Omega_2, (\mathcal{B}_1 \otimes \mathcal{B}_2), \mu \otimes \nu))$. Then $\int f \, d(\mu \otimes \nu) = \int (\int f(x,y) \, d\nu(y)) \, d\mu(x) = \int (\int f(x,y) \, d\nu(x)) \, d\mu(y)$.

(ii) *Show that a random variable with density function* $f(y)$ *has mean* 0 *and variance* 1.

$$\mathbb{E}(y) = \int_{-\infty}^\infty y \, f(y) dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty y \exp\left(-\frac{y^2}{2}\right) dy$$

The antiderivative of $y \exp\left(-\frac{y^2}{2}\right)$ is easily recognized:

$$\frac{d}{dy}\left(-\exp\left(-\frac{y^2}{2}\right)\right) = y \exp\left(-\frac{y^2}{2}\right)$$

Therefore

$$\int_{-\infty}^{\infty} y \exp\left(-\frac{y^2}{2}\right) dy = \left| -\exp\left(-\frac{y^2}{2}\right)\right|_{-\infty}^{\infty} = 0$$

I.e. $\mathbb{E}(y) = 0$.

Since $E(y) = 0$, $\text{Var}(y) = E(y^2)$:

$$\text{Var}(y) = \int_{-\infty}^{\infty} y^2 \, f(y) dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y \, \left(y \exp\left(-\frac{y^2}{2}\right)\right) dy =$$

For this we use partial integration: " $\int_a^b uv' = uv|_a^b - \int_a^b u'v$ " where we set $u = y$ and $v' = y \exp\left(-\frac{y^2}{2}\right)$:

$$= \frac{1}{\sqrt{2\pi}}\left(\left|-y \exp\left(-\frac{y^2}{2}\right)\right|_{-\infty}^{\infty} + \underbrace{\int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy}_{=\sqrt{2\pi} \text{ by (i)}}\right)$$

The first summand in the above expression is zero since $\exp\left(-\frac{y^2}{2}\right)$ vanishes faster as any polynomial as $y^2 \to \infty$:

$$\left|-y \exp\left(-\frac{y^2}{2}\right)\right|_{-\infty}^{\infty} = \lim_{n\to\infty} (-n + n) \exp\left(-\frac{n^2}{2}\right) = 0$$

Hence we have $\text{Var}(y) = \frac{1}{\sqrt{2\pi}}\sqrt{2\pi} = 1$.

Exercise 3.3

Let z_i for $i = 1, \ldots, T$ be independently distributed $N(0, \sigma^2)$. Denote with $\bar{z} = \frac{1}{T}\sum_{t=1}^{T} z_t$ the empirical mean and with $s^2 = \frac{1}{T}\sum_{t=1}^{T}(z_t - \bar{z})^2$ the empirical variance. Show that

$$\frac{Ts^2}{\sigma^2}$$

is X_{T-1} distributed.

From our Q1 Theorem we know that the quadratic form $Q = \frac{z'Az}{\sigma^2}$ is $X_{r,\lambda}$ distributed, with $\lambda = \frac{\mu' A\mu}{\sigma^2}$ and A being a projector with rk(A)=r.

Proof: Since in our example $\mu = 0$ it follows that $\lambda = 0$. Now consider the quadratic form

$$\begin{aligned} \frac{Ts^2}{\sigma^2} &= \frac{T}{\sigma^2}\frac{1}{T}\sum_{t=1}^{T}(z_t - \bar{z})^2 \\ &= \frac{1}{\sigma^2} z'(I_T - \frac{11'}{T})'(I_T - \frac{11'}{T})z \\ &= \frac{z'(I_T - \frac{11'}{T})z}{\sigma^2} \end{aligned}$$

We know from a previous exercise, that $(I_T - \frac{11'}{T})$ is a projector projecting on the orthocomplement of the space spanned by 1. Hence its rank is equal to $T - 1$. So we can apply the Q1 Theorem, whith $A = (I_T - \frac{11'}{T})$, $\lambda = 0$ to get that

$$Q = \frac{z'Az}{\sigma^2}$$

is X_{T-1} distributed.

Exercise 3.4

Show that the expected value (under the true probability measure) of the score is equal to 0, i.e. $\mathbb{E}(s(\theta|y)) = \mathbf{0}$ *and that the variance of the score is equal to the information matrix* $I(\theta|y)$. *Remark: you can assume throughout that you can interchange differentiation and integration when necessary or helpful.*

The random variable is y which is a function from a σ-field in a probability space to $\mathbb{R}$. Moreover any transformation $\mathbb{R} \to \mathbb{R}$ applied on y is still a random variable. The probability measure of $(-\infty, x]$ is thus the function $\int_{-\infty}^{x} f(y)dy$ which satisfies the requirements for a probability measure.

We know

$$s(\theta|y) := \frac{\partial}{\partial\theta}\ell(\theta|y) = \frac{\partial}{\partial\theta}\ln f(y|\theta) = \frac{\frac{\partial}{\partial\theta}f(y|\theta)}{f(y|\theta)}$$

Therefore:

$$\mathbb{E}_\theta\left(s(\theta|y)\right) = \int \frac{\partial}{\partial\theta}\ell(\theta|y)f(y|\theta)dy = \int \frac{\frac{\partial}{\partial\theta}f(y|\theta)}{f(y|\theta)}f(y|\theta)dy =$$
$$= \int \frac{\partial}{\partial\theta}f(y|\theta)dy$$

Here we may change integration and differentiation signs if there exists a function $g(y,\theta_0)$ such that $|\frac{f(y,\theta_0+\delta)-f(y,\theta_0)}{\delta}| \leq g(x,\theta_0)$ and $\int_{\mathbb{R}} g(y,\theta_0)dy \leq \infty$. [1]

$$\mathbb{E}_\theta\left(s(\theta|y)\right) = \int \frac{\partial}{\partial\theta}f(y|\theta)dy = \frac{d}{d\theta}\underbrace{\int 1 f(y|\theta)dy}_{=\mathbb{E}(1)=1} = \frac{d}{d\theta}1 = \mathbf{0}$$

We know $\frac{\partial}{\partial\theta}s(\theta|y) = \frac{\partial}{\partial\theta}\frac{\partial \ln f(y|\theta)}{\partial\theta}$. The score $\frac{\partial \ln f(y|\theta)}{\partial\theta}$ is a vector whose i-th element is:

$$\left[\frac{\frac{\partial f(y|\theta)}{\partial\theta_i}}{f(y|\theta)}\right]_i$$

Differentiating this expression with respect to the vector θ is equivalent to differentiating each element of the score. This results into a matrix, whose i-j-th entry is the following:

$$\left[\frac{\partial}{\partial\theta_j}\left(\frac{\frac{\partial f(y|\theta)}{\partial\theta_i}}{f(y|\theta)}\right)\right]_{i,j} = \left[\frac{\frac{\partial^2 f(y|\theta)}{\partial\theta_j\partial\theta_i}f(y|\theta) - \frac{\partial f(y|\theta)}{\partial\theta_j}\frac{\partial f(y|\theta)}{\partial\theta_i}}{f(y|\theta)^2}\right]_{i,j}$$
$$= \left[\frac{\frac{\partial^2 f(y|\theta)}{\partial\theta_j\partial\theta_i}}{f(y|\theta)} - \underbrace{\frac{\frac{\partial f(y|\theta)}{\partial\theta_j}}{f(y|\theta)}}_{=\text{j-th element of the score}} \frac{\frac{\partial f(y|\theta)}{\partial\theta_i}}{f(y|\theta)}\right]_{i,j}$$

Since the latter two terms are equal to the j-th and the i-th element of the score, respectively, while the very first second derivative is equal the i-j-th entry of the Hessian of f, we may write the matrix whose i-j-th element is given above as:

$$\frac{\partial}{\partial\theta}s(\theta|y) = \frac{1}{f(y|\theta)}\mathbf{H}_f - \mathbf{s}(\theta|y)\mathbf{s}(\theta|y)'$$

[1]compare Casella/Berger (1990): *Statistical Inference*; Duxbury Press, Belmont, CA. p.70

Taking the expected value of the i,j-th element yields the following:

$$\mathbb{E}\left(\left[\frac{\partial}{\partial\theta_j}s_i(\theta|y)\right]_{i,j}\right) = \left[\int\left(\frac{\frac{\partial^2 f(y|\theta)}{\partial\theta_j\partial\theta_i}}{f(y|\theta)} - \mathbf{s}_i(\theta|y)\mathbf{s}_j(\theta|y)'\right)f(y|\theta)dy\right]_{i,j} =$$

$$= \left[\int\frac{\partial^2 f(y|\theta)}{\partial\theta_j\partial\theta_i}dy - \mathbb{E}\left(\mathbf{s}_i(\theta|y)\mathbf{s}_j(\theta|y)'\right)\right]_{i,j}$$

The integral first expression (the second derivative thing) is zero if we may interchange integration and differentiation signs:

$$\int\frac{\partial^2}{\partial\theta_j\partial\theta_i}f(y|\theta)dy = \frac{\partial}{\partial\theta_i}\underbrace{\int\frac{\partial}{\partial\theta_i}f(y|\theta)dy}_{=0(*)} = 0$$

Where we know that expression (*) is zero from the first part of Exercise 3.4.

Knowing that $\frac{\partial}{\partial\theta}s(\theta|y) = \frac{\partial\ell}{\partial\theta\partial\theta'}$, the Hessian of ℓ, we have:

$$\mathbb{E}\left(\frac{\partial\ell}{\partial\theta\partial\theta'}\right) = -\mathbb{E}\left(s(\theta|y)s(\theta|y)'\right)$$

$$\mathbb{E}\left(-H_\ell(\theta|y)\right) = \mathbb{E}\left(s(\theta|y)s(\theta|y)'\right)$$

Therefore the two definitions of the information $I(\theta|y)$ yield the same result.

Exercise 3.5

proof of Lemma 2:

$$F = \frac{\frac{(\widehat{\beta}-\widetilde{\beta})'(X'X)(\widehat{\beta}-\widetilde{\beta})}{\sigma^2}(T-k)}{\frac{\widehat{u}'\widehat{u}}{\sigma^2}m} \quad \text{is } F_{m,T-k}.$$

The denominator $\frac{\widehat{u}'\widehat{u}}{\sigma^2}$ is χ^2_{T-k} by theorem Q4.

we know that

$$\widehat{\beta}-\widetilde{\beta} = (X'X)^{-1}R'\left[R(X'X)^{-1}R'\right]^{-1}(R\widehat{\beta}-r).$$

so we can rearrange:

$$
\begin{aligned}
&(\widehat{\beta}-\widetilde{\beta})'(X'X)(\widehat{\beta}-\widetilde{\beta}) = \\
&= (R\widehat{\beta}-r)'\left[R(X'X)^{-1}R'\right]^{-1}R(X'X)^{-1}X'X(X'X)^{-1}R'\left[R(X'X)^{-1}R'\right]^{-1}(R\widehat{\beta}-r) \\
&= (R\widehat{\beta}-r)'\left[R(X'X)^{-1}R'\right]^{-1}(R\widehat{\beta}-r)
\end{aligned}
$$

null hypothesis holds: $R\beta = r$

we know that $\widehat{\beta} = (X'X)^{-1}X'y = (X'X)^{-1}X'(X\beta+u) = \beta + (X'X)^{-1}X'u$. So $(R\widehat{\beta}-r) = R\beta - r + R(X'X)^{-1}X'u$ (as $R\beta - r = 0$ by assumption).

Plugging in above yields:

$$
(R\widehat{\beta}-r)'\left[R(X'X)^{-1}R'\right]^{-1}(R\widehat{\beta}-r) = u'X(X'X)^{-1}R'\left[R(X'X)^{-1}R'\right]^{-1}R(X'X)^{-1}X'u.
$$

Note that the matrix "between the u" is symmetric and idempotent and has Rank m (= number of restrictions = rank of R). So by Theorem Q1, this quantity (divided by σ^2) is χ^2_m (as the u are centered).

As $\widehat{u} = (I-P)y = (I-P)u$ (where $P = X(X'X)^{-1}X'$) we can write:

$$
\begin{aligned}
F &= \frac{(\widehat{\beta}-\widetilde{\beta})'(X'X)(\widehat{\beta}-\widetilde{\beta}) \quad (T-k)}{\widehat{u}'\widehat{u} \quad m} \\
&= \frac{u'X(X'X)^{-1}R'\left[R(X'X)^{-1}R'\right]^{-1}R(X'X)^{-1}X'u \quad (T-k)}{u'(I-P)u \quad m}
\end{aligned}
$$

What's left to check for proving that the whole quantity is $F_{m,T-k}$ is the independence of the denominator and the nominator. Note that

$$
(I-P)X(X'X)^{-1}\ldots = \left(X - X(X'X)^{-1}X'X\right)(X'X)^{-1}\ldots = 0
$$

By theorem Q3, this establishes independence.

$\beta = \beta^0$**, where** $R\beta^0 \neq r$

if this is the case, we know from class (theorem Q5 and the stuff before) that $\widehat{\beta} - \widetilde{\beta} = \beta^0 + (X^+)u - \widetilde{\beta} = (X^+)\left[u + X(\beta^0 - \widetilde{\beta})\right]$.

$u + X(\beta^0 - \widetilde{\beta})$ is $N(X(\beta^0 - \widetilde{\beta}), \sigma^2 I_T)$.

Caution: This conclusion is incorrect: We did the correct thing in class.

$$
\begin{aligned}
F &= \frac{(\widehat{\beta} - \widetilde{\beta})'(X'X)(\widehat{\beta} - \widetilde{\beta}) \quad (T-k)}{\widehat{u}'\widehat{u} \quad m} \\
&= \frac{(u + X(\beta^0 - \widetilde{\beta}))'X(X'X)^{-1}(X'X)(X'X)^{-1}X'(u + X(\beta^0 - \widetilde{\beta})) \quad (T-k)}{u'(I-P)u \quad m} \\
&= \frac{(u + X(\beta^0 - \widetilde{\beta}))'X(X'X)^{-1}(X'X)(X'X)^{-1}X'(u + X(\beta^0 - \widetilde{\beta})) \quad (T-k)}{u'(I-P)u \quad m} \\
&= \frac{(u + X(\beta^0 - \widetilde{\beta}))'X(X'X)^{-1}X'(u + X(\beta^0 - \widetilde{\beta})) \quad (T-k)}{u'(I-P)u \quad m}
\end{aligned}
$$

By theorem Q1, the nominator (divided by σ^2) is χ_k^λ as $X(X'X)^{-1}X'$ has rank k and is symmetric and idempotent. $\lambda = \frac{(\beta^0 - \widetilde{\beta})'X'X(\beta^0 - \widetilde{\beta})}{\sigma^2}$.

Note that $(I-P)X(X'X)^{-1}X' = 0$. So by theorem Q3, denominator and nominator are independent.

Exercise 3.6

leave out $T-k$ and m

$$
\frac{\widetilde{u}'\widetilde{u} - \widehat{u}'\widehat{u}}{\widehat{u}'\widehat{u}} = \frac{\widetilde{u}'\widetilde{u}}{\widehat{u}'\widehat{u}} - 1 = \frac{\widetilde{u}'\widetilde{u}/T}{\widehat{u}'\widehat{u}/T} - 1 = \frac{1 - \widetilde{R}^2}{1 - R^2} - 1 = \frac{1 - \widetilde{R}^2 - (1 - R^2)}{1 - R^2} = \frac{R^2 - \widetilde{R}^2}{1 - R^2}
$$

Exercise 3.7

Exercise 3.7 is very similar to Exercise 3.5.

Exercise 3.8

Denote with $\hat{\beta}$ *the unrestricted OLS estimator and with* $\tilde{\beta}$ *the restricted OLS estimator under the restriction* $R\beta = r$*. Then the quantities* $(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)$ *and* $(\hat{\beta} - \tilde{\beta})'X'X(\hat{\beta} - \tilde{\beta})$ *are equal (and can be written, by Lemma 3, as* $\tilde{u}'\tilde{u} - \hat{u}'\hat{u}$*).*

$\hat{\beta} = (X'X)^{-1}X'y$, $\tilde{\beta} = \hat{\beta} - Q(R\hat{\beta} - r)$ with $Q = (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}$. Thus $(\hat{\beta} - \tilde{\beta}) = Q(R\hat{\beta} - r)$ and $(\hat{\beta} - \tilde{\beta})'X'X(\hat{\beta} - \tilde{\beta}) = (R\hat{\beta} - r)'Q'X'XQ(R\hat{\beta} - r)$.

Now

$$Q' = \left[(X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}\right]' = \left[[R(X'X)^{-1}R']'\right]^{-1}\left[(X'X)^{-1}R'\right]' =$$

$$= \left[R[R(X'X)^{-1}]'\right]^{-1}R(X'X)^{-1} = \left[R(X'X)^{-1}R'\right]^{-1}R(X'X)^{-1}$$

Thus

$$(\hat{\beta} - \tilde{\beta})'X'X(\hat{\beta} - \tilde{\beta}) =$$

$$= (R\hat{\beta} - r)'\left[R(X'X)^{-1}R'\right]^{-1}R(X'X)^{-1}X'X(X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}(R\hat{\beta} - r) =$$

$$= (R\hat{\beta} - r)'\left[R(X'X)^{-1}R'\right]^{-1}\left[R(X'X)^{-1}R'\right]\left[R(X'X)^{-1}R'\right]^{-1}(R\hat{\beta} - r) =$$

$$= (R\hat{\beta} - r)'\left[R(X'X)^{-1}R'\right]^{-1}(R\hat{\beta} - r)$$

QED

Exercise 3.9

Show Lemma 7

$(y - X_2\beta_2^*)'M_1(y - X_2\beta_2^*) = (y - X\hat{\beta})'(y - X\hat{\beta}) + (\hat{\beta}_2 - \beta_2^*)'H(\hat{\beta}_2 - \beta_2^*)$

Proof:

Let $y - X_2\beta_2^* = (y - x\hat{\beta}) + (x\hat{\beta} - X_2\beta_2^*) = (y - x\hat{\beta}) + x_1\hat{\beta}_1 + x_2(\hat{\beta}_2 - \beta_2^*)$, then we have

$(y - X_2\beta_2^*)' M_1 (y - X_2\beta_2^*) = [(y - x\hat{\beta}) + x_1\hat{\beta}_1 + x_2(\hat{\beta}_2 - \beta_2^*)]' M_1 [(y - x\hat{\beta}) + x_1\hat{\beta}_1 + x_2(\hat{\beta}_2 - \beta_2^*)] = \underbrace{(y - x\hat{\beta})'}_{\hat{u}} M_1 (y - x\hat{\beta}) + 2(y - x\hat{\beta})' \underbrace{M_1 X_1}_{=0} \hat{\beta}_1 + 2 \overbrace{\underbrace{(y - x\hat{\beta})' M_1}_{\hat{u}'} X_2}^{=0} (\hat{\beta}_2 - \beta_2^*) + 2\hat{\beta}_1' \underbrace{X_1' M_1}_{=0} X_2 (\hat{\beta}_2 - \beta_2^*) + (\hat{\beta}_2 - \beta_2^*)' \underbrace{X_2' M_1 X_2}_{H} (\hat{\beta}_2 - \beta_2^*)$

We know that M_1 is orthogonal to the space spanned by col(X_1), thus $M_1 X_1 = 0$ and

$\hat{u} \in [col(X_1, X_2)]^{\perp} \subset [col(X_1)]^{\perp}$, so

$\hat{u}' M_1 = \hat{u}'$.

Similarly, $\hat{u} \in [col(X_2)]^{\perp}$, so $\hat{u}' X_2 = 0$.

Hence the right side of the equation above only remains

$(y - X\hat{\beta})'(y - X\hat{\beta}) + (\hat{\beta}_2 - \beta_2^*)' H (\hat{\beta}_2 - \beta_2^*)$, which is just what we have to show.

Exercise 3.10, first part

(i) Show Lemma 8 of the document 'Some Basics on Testing':

Under the null hypothesis that $\beta_2 = \beta_2^*$ (with $\beta_2^* \in \mathbb{R}^{k_2}$), the quantity

$$F = \frac{(\hat{\beta}_2 - \beta_2^*)' H (\hat{\beta}_2 - \beta_2^*)}{\hat{u}'\hat{u}} \frac{T - k}{k_2}$$

is $F_{k_2, T-k}$ distributed.

By the Frisch-Waugh Theorem

$\hat{\beta}_2 = (X_2' M_1 X_2)^{-1} X_2' M_1 y$

Under the Null-Hypothesis, we hence receive

$Y = \tilde{X}_2 \beta_2^* + u$

Inserting this into $\hat{\beta}_2$ yields

$$\begin{aligned}\hat{\beta}_2 &= (X_2' M_1 X_2)^{-1}(X_2' M_1 X_2)\beta_2^* + (X_2' M_1 X_2)^{-1} X_2' M_1 u \\ &= \beta_2^* + (X_2' M_1 X_2)^{-1} X_2' M_1 u\end{aligned}$$

We also know that

$$H = X_2' M_1 X_2$$

Inserting these into the nominator above yields

$$u'((X_2' M_1 X_2)^{-1} X_2' M_1)' \underbrace{X_2' M_1 X_2 (X_2' M_1 X_2)^{-1}}_{I} X_2' M_1 u$$

$$u' \underbrace{M_1' X_2 (X_2' M_1 X_2)^{-1} X_2' M_1}_{symmetric+idempotent} u$$

The rank of this projector is k_2. By Theorem Q1 and exercise 3.5 we hence know that this, if divided by σ^2, is distributed as χ^2_m. We also again know by Theorem Q4 that the denominator is, if divided by σ^2, distributed as χ^2_{T-k}.

So what remains is to show independence by multiplying both projectors and receiving:

$$\underbrace{(I - \tilde{X}_2 \tilde{X}_2{}^+)}_{(I-P_2)} \underbrace{\tilde{X}_2 \tilde{X}_2{}^+}_{idempotent} = \tilde{X}_2 \tilde{X}_2{}^+ - \tilde{X}_2 \tilde{X}_2{}^+ = 0$$

Again by Theorem Q3 this establishes independence.

(ii) Derive the distribution of the quantity F when the ture value of β_2 is equal to some arbitrary, fixed β_2^0.

Exercise 3.11

Exercise 3.11 is concerned with MATLAB programming in groups of two.