Solutions for Econometrics I Homework No.4

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These Exercises were not typed in LaTeX, but instead these are scanned handwritten solutions: The nice handwriting is by Michael Pichler, the "less nice" one by Lukas Reiss.

4.1

$H_0: \sigma^2 = \sigma_0^2$

$H_1: \sigma^2 \neq \sigma_0^2$ one-sided also possible

size: α

$\hat{\sigma}^2 = \frac{\hat{u}'\hat{u}}{T-k}$ $\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1})$

$(T-k)\frac{\hat{\sigma}^2}{\sigma_0^2} \sim \chi^2_{T-k}$ under H_0

$$\frac{(T-k)\sigma_0^2}{\chi^2_{(\frac{\alpha}{2}, T-k)}} > \hat{\sigma}^2 > \frac{(T-k)\sigma_0^2}{\chi^2_{(1-\frac{\alpha}{2}, T-k)}}$$

~~4.4~~

$$\cancel{(T-k)\frac{\hat{\sigma}^2}{\sigma_0^2} > \chi^2_{(1-\frac{\alpha}{2}, T-k)}}$$

$$\cancel{\hat{\sigma}^2 >}$$

$$\chi^2_{\frac{\alpha}{2}} < (T-k)\frac{\hat{\sigma}^2}{\sigma_0^2} < \chi^2_{(1-\frac{\alpha}{2})} \quad | : (T-k) \cdot \sigma_0^2$$

$$\frac{\sigma_0^2}{T-k}\chi^2_{\frac{\alpha}{2}} < \hat{\sigma}^2 < \chi^2_{(1-\frac{\alpha}{2})}\frac{\sigma_0^2}{T-k}$$

4.3 $y_t = x_t \beta + u_t$

$$\sigma^2 \Omega = \sigma^2 \underbrace{\begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & T \end{pmatrix}}_{R'R} \qquad X^* = PX$$

$$R = \begin{pmatrix} 1 & & & \\ & \sqrt{2} & & \\ & & \ddots & \\ & & & \sqrt{T} \end{pmatrix}, \quad R^{-1} = \begin{pmatrix} 1 & & & \\ & \frac{1}{\sqrt{2}} & & \\ & & \ddots & \\ & & & \frac{1}{\sqrt{T}} \end{pmatrix} = P$$

$$\hat\beta_T = \frac{x'y}{x'x}$$

$$\tilde\beta_T = \frac{\sum \left(x_t \frac{1}{\sqrt{t}} y_t\right)}{\sum x_t^2 \frac{1}{\sqrt{t}}} = (x' \Omega^{-1} X)^{-1} (x' \Omega^{-1} y)$$

(i) $\underline{Var(\tilde\beta_T)} = \sigma^2 \frac{1}{\sum \frac{x_t^2}{t}} \xrightarrow{T\to\infty} 0$ consistent

$\sigma^2 (x' \Omega^{-1} X)^{-1}$

$$\sum \frac{x_t^2}{t} \geq c_1^2 \cdot \sum \frac{1}{t}$$

$\check\beta = Dy + u$

$Var(\check\beta) = D\, E(u'u)\, D'$

$$Var(\hat\beta) = \sigma^2 (x'X)^{-1} X' \Omega X (x'X)^{-1} \qquad (\hat\beta = (x'X)^{-1} x'y)$$

$$= \sigma^2 \frac{\sum x_t^2 t}{\left(\sum x_t^2\right)^2} \quad \left[\begin{array}{l} \geq c_1^2 \sum t = c_1^2 \frac{T(T-1)}{2} \\ \leq (c_2^2 T)^2 \end{array} \right. \geq \overbrace{\frac{\sigma^2}{2} \frac{c_1^2}{c_2^4}}^{c} \frac{T^2 - T}{T^2} = c(1-$$

4.4

(i) $(u_t)_{t\in\mathbb{Z}}$

$$\gamma(s) = E(u_t - Eu_t)(u_{t-s} - Eu_{t-s})$$

$\vdots$

$$|a \cdot b| \leq \underbrace{\|a\|}_{\sqrt{a\cdot a}} \cdot \|b\|$$

$$|E[a \cdot b]| \leq \sqrt{Ea^2}\sqrt{Eb^2}$$

$$|E[u_t u_{t-s}]| \leq \sqrt{Eu_t^2}\sqrt{Eu_{t-s}^2}$$

$\Downarrow$

$$\underbrace{|E[(u_t-\mu)(u_{t-s}-\mu)]|}_{|\gamma(s)|} \leq \underbrace{\sqrt{E(u_t-\mu)^2}\sqrt{E(u_{t-s}-\mu)^2}}_{\gamma(0)}$$

(ii)

MA(m)-process $\qquad \varepsilon_t: WN$

$$u_t = \varepsilon_t + \theta_1\varepsilon_{t-1} + \ldots\ldots + \theta_m\varepsilon_{t-m}$$

$$\cancel{\gamma(s) = E(u_t u_{t-s}) = E[\varepsilon_t + \theta_1}$$

$$Var(u_t) = (1 + \theta_1^2 + \ldots\ldots + \theta_m^2)\sigma_\varepsilon^2$$

4.4) – cont.

$\gamma(s)$ only a function of the lag

$$\gamma(s,t) \overset{s \leq m}{=} \operatorname{cov}[u_t, u_{t-s}] = \mathbb{E}[(\varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_m \varepsilon_{t-m})(\varepsilon_{t-s} + \theta_1 \varepsilon \ldots + \ldots$$

$$= \mathbb{E}[\varepsilon_t(\ldots\ldots) + \theta_1 \varepsilon_{t-1}(\ldots) + \theta_s \varepsilon_{t-s}(\ldots \varepsilon_{t-s} \ldots) +$$

$$+ \theta_{s+1} \varepsilon_{t-s-1}(\ldots \theta_1 \varepsilon_{t-s-1} \ldots) + \ldots + \theta_m \varepsilon_{t-m}(\ldots \theta_{m-s} \varepsilon_{t-s-m} \ldots$$

$$= \theta_s \sigma_\varepsilon^2 + \theta_{s+1} \theta_1 \sigma_\varepsilon^2 + \ldots + \theta_m \theta_{m-s} \sigma_\varepsilon^2$$

$$= (\theta_s + \theta_{s+1}\theta_1 + \ldots + \theta_m \theta_{m-s}) \sigma_\varepsilon^2$$

$$E\Big[\Big(\sum_{i=0}^{m} \theta_i \varepsilon_{t-i}\Big)\underbrace{\Big(\sum_{i=0}^{m} \theta_i \varepsilon_{t-s-i}\Big)}_{\sum_{j=s}^{m+s} \theta_{j-s} \varepsilon_{t-j}}\Big]$$

$$= \mathbb{E}\Big[\sum_{i=0}^{m} \sum_{j=s}^{m+s} \theta_i \theta_{j-s} \varepsilon_{t-i} \varepsilon_{t-j}\Big]$$

$$= \sigma_\varepsilon^2 \Big[\sum_{k=s}^{m} \theta_k \theta_{k-s}\Big] = \gamma(s) \quad \text{independent of } t$$

$\rightarrow$ MA – Processes are stationary

4.4

(iii)

$u_t = a_1 u_{t-1} + a_2 u_{t-2} + \varepsilon_t$

$\underbrace{(1 - a_1 B - a_2 B^2)}_{=: a(B)} u_t = \varepsilon_t$

→ ~~root~~ stationary if roots outside unit circle

roots of $a(z)$:

$$\frac{1}{q_{1,2}} = \frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{-2a_2}$$

$$\left(\frac{1}{1-a_1 z}\right)\left(\frac{1}{1-a_2 z}\right)$$

roots shouldn't have absolute value 1

$$a^{-1}(z) = \sum_{j=-\infty}^{\infty} k_j z^j$$

$a^{-1}(z) \cdot a(z) = 1$

$$\rightarrow \sum_{j=-\infty}^{\infty} k_j z^j (1 - a_1 z - a_2 z^2) = 1$$

~~z^0: $k_0 + k_{-1}$~~

z^0: $k_0 = 1$

z^1: $-a_1 k_0 + k_1 = 0 \quad \rightarrow k_1 = a_1$

z^2: $-a_2 k_0 + k_1 a_1 + k_2 = 0 \quad \rightarrow k_2 = -a_1^2 + a_2$

z^3: $-a_2 k_1 + a_1 k_2 + k_3 = 0 \quad \rightarrow k_3 = -a_1 + a_2 k_1$

4.4 -cont.

if we know

$$\sum a_k z^k = \sum b_k z^k$$ and both converge

can set $a_k = b_k$

$$\sum_{j=0}^{\infty} k_j z^j (1 - a_1 z - a_2 z^2) = 1 + 0 \cdot z + 0 \cdot z^2 + 0 \ldots$$

assume stationary, causal solution

$$\gamma(s) = Cov(u_t, u_{t-s}) = E u_t u_{t-s}$$

$$u_t = a_1 u_{t-1} + a_2 u_{t-2} + \varepsilon_t$$

$$u_{t-s} u_t = a_1 u_{t-1} u_{t-s} + a_2 u_{t-2} u_{t-s} + u_{t-s} \varepsilon_t$$

$s=0$, E: $$E(u_t^2) = a_1 E(u_t u_{t-1}) + a_2 E(u_t u_{t-2}) + E(\varepsilon_t u_t)$$

$$\rightarrow \gamma(0) = a_1 \gamma(1) + a_2 \gamma(2) + \sigma_\varepsilon^2$$

$s=1$: $$\gamma(1) = a_1 \gamma(0) + a_2 \gamma(1)$$

$$\gamma(s) = a_1 \gamma(s-1) + a_2 \gamma(s-2)$$

Yule-Walker equations

one-to-one correspondence between autocovariance and coefficients

4.5)

$(\varepsilon_t)_{t\in\mathbb{Z}}$ WN

$E(\varepsilon_t)=0$, $Var(\sigma_\varepsilon^2)\ \forall t$

$Cov(\varepsilon_t, \varepsilon_{t+\tau})=0 \quad \forall\tau\neq 0\ \forall t$

Let $\{\varepsilon_{st}\}_{t\in\mathbb{Z}}$ ~~$s\neq 1$~~ $s\in\mathbb{N}$

$\{\dots\ \varepsilon_{-2s}, \varepsilon_{-s}, \varepsilon_0, \varepsilon_s, \varepsilon_{2s}, \dots\}$

if ~~$s\neq 0$~~ $Var(\varepsilon_{st})=\sigma_\varepsilon^2$

$E(\varepsilon_{st})=0$

$Cov(\varepsilon_{st}, \varepsilon_{s(t+\tau)})=0 \quad \forall\tau\neq 0$

$\{x_t\}_{t\in\mathbb{Z}} \quad x_t=\varepsilon_t+\varepsilon_{st} \qquad s\neq 1 \qquad \tau\neq 0\ ,\ \tau>0$

$Cov(x_t, x_{t+\tau})=\mathbb{E}(\varepsilon_t+\varepsilon_{st})(\varepsilon_{t+\tau}+\varepsilon_{st+s\tau})=$

$=\mathbb{E}(\underbrace{\varepsilon_t\varepsilon_{t+\tau}}_{0 \text{ as } \tau\neq 0})+\mathbb{E}(\underbrace{\varepsilon_t\varepsilon_{st+s\tau}}_{\cancel{0 \text{ as } s\neq 1}\ \text{if } \tau=t(s-1)})+\mathbb{E}(\varepsilon_{st}\varepsilon_{t+\tau})+\mathbb{E}(\underbrace{\varepsilon_{st}\varepsilon_{st+s\tau}}_{0 \text{ as } \tau\neq 0})$

~~variance still constant~~

Exercise 4.6

show that $\hat{\sigma}^2 = \frac{1}{T-K} \hat{u}'\hat{u}$ is consistent :

hence :

$\text{plim}\, \hat{\sigma}^2 = \sigma^2$

$$\hat{\sigma}^2 = \frac{\hat{u}'\hat{u}}{T-K} = \frac{T}{T-K} \frac{u'(I-P)u}{T} = \frac{T}{T-K}\left(\frac{u'u}{T} - \frac{u'X}{T}\left(\frac{X'X}{T}\right)^{-1}\frac{X'u}{T}\right)$$

$$\text{plim}\, \hat{\sigma}^2 = \underbrace{(p)\lim \frac{T}{T-K}}_{1}\left(\text{plim}\frac{u'u}{T} - \underbrace{\text{plim}\frac{u'X}{T}}_{0} \cdot \underbrace{(p)\lim\left(\frac{X'X}{T}\right)^{-1}}_{(D1):\ \Pi_{XX}^{-1}\ \exists + \text{finite}} \underbrace{\text{plim}\frac{X'u}{T}}_{0}\right)$$

$\left[\lim x_n \to x,\ \lim y_n \to y \Rightarrow \lim(x_n y_n) = \lim(x_n) \cdot \lim(y_n)\right]$

as:

$$\text{plim}\frac{X'u}{T} = \lim_{T\to\infty} P\left(\left\|\frac{X'u}{T}\right\| > \varepsilon\right) = 0$$

$$\lim_{T\to\infty} P\left(\frac{1}{T}\|X'u\| > \varepsilon\right) = 0$$

$$(D4):\ \|X\| < \infty;\ \lim_{T\to\infty} P\left(\|u\| > T\boxed{\frac{\varepsilon}{\|X\|}}\right) = 0$$

(Claim: $\left[\|X'u\| \leq \|X'\|\,\|u\|\right]$

So:

$$\text{plim}\, \hat{\sigma}^2 = \text{plim}\frac{u'u}{T} = \sigma^2\ ?$$

use Chebyshev's Inequality:

$$P(|X-\mu| > \varepsilon) \leq \frac{\sigma_x^2}{\varepsilon^2}$$

$$\boxed{P\left(\left|\frac{u'u}{T} - \sigma^2\right| > \varepsilon\right) \leq \frac{\square}{\varepsilon^2} \leq \frac{\frac{2\sigma^4}{T}}{\varepsilon^2} \xrightarrow{T\to\infty} 0}$$

$$\square\ \text{Var}\left(\frac{u'u}{T}\right) = \text{Var}\left(\frac{1}{T}\sum_{t=1}^{T} u_t^2\right) = \frac{1}{T^2}\text{Var}\left(\sum_{t=1}^{T} u_t^2\right) = \frac{1}{T^2}\sum_{t=1}^{T}\text{Var}(u_t^2)$$

$$= \frac{T}{T^2}\, 2\sigma^4$$

$$\boxed{= \frac{2\sigma^4}{T}}$$

$$\text{Var}(u_t^2) = E\left((u_t^2)^2\right) - \left(E(u_t^2)\right)^2$$

$$\text{Var}(u_t^2) = E(u_t^4) - (\sigma^2)^2$$

$$(B3) = 3\sigma^4 - \sigma^4$$

$$\text{Var}(u_t^2) = 2\sigma^4$$