Solutions for Probability and Statistics

IHS PIE & VGSF, classes of 2005, Vienna

Corrected by Prof. Helmut Strasser, WU Wien

February 15, 2006

Abstract

These are the solutions for the Feb 5th version of Prof.Strasser's sample test on *http://helmut.strasserweb.net/public/vgsf.html#Anchor-Midter-5542* .
If you notice any error, please email zeugner@ihs.ac.at with a corrected LaTeX file (LaTeX files are to be found on the website
http://elaine.ihs.ac.at/∼zeugner/stats/).
The respective identifiers for those examples in Prof. Strasser's lecture notes from Feb 1st are given in italic letters. For the numberings referring to other versions of Prof. Strasser's notes, please consult the table at the next page.
The sign ✓ indicates that the respective solution has been approved by Prof. Strasser. The sign ✗ indicates that Prof. Strasser has comments on the exercise (printed at the beginning of each such exercise).

Table 1: Exercise Number Reference

Test 2006-02-05	Notes 1 2006-01-05	Notes 3 2006-01-25	Notes 4 2006-01-31	Notes 5 2006-02-01	Feedback	page
E1	1.4a	1.4	1.4	1.4	✓	p. 4
E2	1.4b	1.5	1.5	1.5	✓	p. 4
E3	1.7a	1.8	1.8	1.8	✓	p. 5
E4	1.14	1.16	1.16	1.16	✓	p. 6
E5	1.16?	1.18	1.18	1.18	✓	p. 6
E6a	1.21	1.23	1.23	1.23	✓	p. 7
E6b	1.22a	1.24a	1.24a	1.24a	✗	p. 8
E7	1.24?	1.30	1.30	1.30	✗	p. 9
E8	2.3	2.3	2.3	2.3	✗	p. 9
E9	na	2.4	2.4	2.4	✗	p. 10
E10	na	2.5	2.5	2.5	✗	p. 11
E11	na	2.22	2.22	2.22	✗	p. 12
E12	na	3.6	3.6	3.6	✓	p. 13
E13	na	3.7	3.7	3.7	✗	p. 14
E14	3.14	3.18	3.18	3.18	✓	p. 15
E15	3.20	3.24	3.24	3.24	✓	p. 15
E16	na	4.5	4.5	4.5	✓	p. 16
E17	na	4.9	4.9	4.9	✓	p. 17
E18	na	4.14	4.14	4.14	✓	p. 18
E19	na	5.2a	5.2a	5.2a	✓	p. 19
E20	na	5.2b	5.2b	5.2b	✗	p. 19
E21	5.5	6.5	6.5	6.3	✗	p. 20
E22	5.4	6.4	6.4	6.4a	✓	p. 21
E23	na	na	na	6.11	✓	p. 21
E24	6.10	7.12	7.12	7.13	✗	p. 22
E25	6.11	7.13	7.13	7.14	✓	p. 23
E26	na	na	7.18	7.19	✓	p. 24
E27	6.15	7.17	7.19	7.20	✓	p. 25
E28	na	na	7.32	7.35	✗	p. 26
E29	na	na	7.34	7.37	✓	p. 27
E30	7.4	8.4	8.4	8.4	✓	p. 28
E31	see Thm 7.16	see Thm 8.16	see Thm 8.16	see Thm 8.18		p. 28
E32	see Thm 7.17	see Thm 8.18	see Thm 8.18	see Thm 8.18	✓	p. 30
E33	7.27	8.28	8.28	8.30	✓	p. 31
E34	7.33	8.33	8.33	8.35	✓	p. 32
E35	7.34b	8.34b	8.34b	8.36b	✓	p. 33
I1	1.7b	1.9	1.9	1.9	✓	p. 35
I2	1.15	1.17	1.17	1.17	✓	p. 37
I3	1.19	1.21	1.21	1.21	✓	p. 38
I4	2.5	2.7	2.7	2.7	✓	p. 40
I5	2.9	2.11	2.11	2.11	✓	p. 40
I6	3.10	3.14	3.14	3.14	✓	p. 41

Table 2: Exercise Number Reference (continued)

Test 2006-02-05	Notes 1 2006-01-05	Notes 3 2006-01-25	Notes 4 2006-01-31	Notes 5 2006-02-01	Feedback	page
I7	3.11	3.15	3.15	3.15	✓	p. 42
I8	3.22b	3.26b	3.26b	3.26b	✓	p. 43
I9	3.25	3.31	3.31	3.32	✓	p. 44
I10	na	4.3a	4.3a	4.3a	✗	p. 44
I11	na	4.12	4.12	4.12	✗	p. 45
I13	na	4.13	4.13	4.13	✗	p. 46
I14	na	4.15	4.15	4.15	✗	p. 48
I15	na	na	na	6.8	✓	p. 49
I16	na	na	na	6.10	✓	p. 50
I17	6.9	7.11	7.11	7.12	✓	p. 51
I18	na	na	7.33	7.36	✓	p. 52
I19	7.5	8.5	8.5	8.5	✓	p. 53
I20	7.11	8.11	8.11	8.12	✓	p. 54
I21	7.25	8.25	8.25	8.27	✓	p. 55
I22	7.30a	8.30a	8.30a	8.32a	✓	p. 56
I23	7.35	8.35	8.35	8.37	✓	p. 57
I24	7.36	8.36	8.36	8.38	✓	p. 58
A1	1.8?	1.10	1.10	1.10	✓	p. 60
A2	1.25	1.28	1.28	1.28	✓	p. 61
A3	1.36 a)	1.40a	1.40a	1.40a	✓	p. 62
A4	1.36 b)	1.40b	1.40b	1.40b	✓	p. 63
A5	2.7	2.9	2.9	2.9	✓	p. 64
A6	3.8	3.12	3.12	3.12	✓	p. 65
A7	6.4	7.5	7.5	7.6	✓	p. 66
A8	na	7.7	7.7	7.8		
A9	7.9	8.9	8.9	8.10	✓	p. 67
R1	1.17	1.19	1.19	1.19		p. 69
R2	1.32	1.36	1.36	1.36		p. 69
R3	1.33	1.37	1.37	1.37		p. 70
R4	1.47	1.44	1.44	1.44		p. 73
R5	2.8	2.10	2.10	2.10		p. 74
R6	2.19	2.20	2.20	2.20		p. 75
R7	na	2.27	2.27	2.27		p. 76
R8	na	na	na	3.30		p. 77
R9	na	na	na	3.34		
R10	na	na	na	6.12		p. 77
R11	na	na	na	7.4		p. 79
R12	na	na	na	7.22		p. 80
R13	na	na	na	7.30		p. 82
R14	na	na	na	8.8		p. 82
R15	na	na	na	8.13		p. 83
R16	na	na	na	8.40		

1 Easy Questions

Easy 1 (Test Feb-05), *1.4 (Notes Feb-01)* ✓

PROBLEM: Let $\mu|\mathcal{A}$ be a content. Then $A_1 \subseteq A_2$ implies $\mu(A_1) \leq \mu(A_2)$.

——

Let $A_1, A_2 \in \mathcal{A}$. A_1. Then $A_2 \setminus A_1$ are disjoint. Using the properties of $\mu|\mathcal{A}$ a content we get;

$$\mu(A_1) \leq \mu(A_1) + \mu(A_2 \setminus A_1) = \mu(A_1 \cup (A_2 \setminus A_1)) = \mu(A_2)$$

Easy 2 (Test Feb-05), *1.5 (Notes Feb-01)* ✓

PROBLEM: Show that every content satisfies the inclusion-exclusion law:

$$\mu(A_1) + \mu(A_2) = \mu(A_1 \cup A_2) + \mu(A_1 \cap A_2)$$

——

We know:

$$\mu(A_1) = \mu(A_1 \setminus (A_1 \cap A_2)) + \mu(A_1 \cap A_2)$$
$$\mu(A_2) = \mu(A_2 \setminus (A_1 \cap A_2)) + \mu(A_1 \cap A_2)$$

Adding both lines and using additivity of the content we get

$$\begin{aligned}
\mu(A_1) + \mu(A_2) &= \mu(A_1 \setminus (A_1 \cap A_2)) + \mu(A_1 \cap A_2) + \mu(A_2 \setminus (A_1 \cap A_2)) + \mu(A_1 \cap A_2) \\
&= \mu((A_1 \setminus (A_1 \cap A_2)) \cup (A_1 \cap A_2) \cup (A_2 \setminus (A_1 \cap A_2))) + \mu(A_1 \cap A_2) \\
&= \mu((A_1 \cup A_2))) + \mu(A_1 \cap A_2)
\end{aligned}$$

Easy 3 (Test Feb-05), *1.8 (Notes Feb-01)* ✓

PROBLEM: Let $\Omega = (-\infty, \infty]$ and let $\mathcal{R}$ be the system of subsets arising as unions of finitely many intervals of the form $(a, b]$ where $-\infty \leq a < b \leq \infty$ (left-open and right-closed intervals). Explain why $\mathcal{R}$ is a field. (Include $\emptyset$ as the union of nothing).

—–

take $(\Omega, \mathcal{R})$ s.t. $\Omega = (\infty, \infty]$ and $\forall R \in \mathcal{R}: \ R = \bigcup_{i=1}^{n}(a_i, b_i] \quad \text{with} -\infty \leq a_i < b_i \leq \infty$

$\forall R \in \mathcal{R}$ as defined above, we have $R \in \Omega$

further, $\mathcal{R}$ satisfies the 3 properties of a field:

1. $\Omega = (-\infty, a] \cup (a, \infty] \quad \Rightarrow \quad \Omega \in \mathcal{R}$

   $\phi =$ union of nothing $\quad \Rightarrow \quad \phi \in \mathcal{R}$

2. take $R_1, R_2 \in \mathcal{R}$, with $R_1 = \bigcup_{i=1}^{n}(a_i, b_i]$ and $R_2 = \bigcup_{j=1}^{m}(c_j, d_j]$

   $R_1 \cup R_2 = (\bigcup_{i=1}^{n}(a_i, b_i]) \cup \left(\bigcup_{j=1}^{m}(c_j, d_j]\right) \quad \Rightarrow \quad R_1 \cup R_2 \in \mathcal{R}$ (as this is again a finite union of intervals)

   $R_1 \cap R_2 = (\bigcup_{i=1}^{n}(a_i, b_i]) \cap \left(\bigcup_{j=1}^{m}(c_j, d_j]\right) = \bigcup_{i,j}[(a_i, b_i] \cap (c_j, d_j]]$. This follows from the distributive law. So $R_1 \cap R_2$ is again a finite union of intervals of the form $(a, b] \quad \Rightarrow \quad R_1 \cap R_2 \in \mathcal{R}$

3. take $R_1 = \bigcup_{i=1}^{n}(a_i, b_i]$, then $R_1^c = \bigcap_{i=1}^{n}(a_i, b_i]^c$. Now for every $A_i = (a_i, b_i]$ we jave $A_i^c = (-\infty, a_i] \cup (b_i, \infty]$, thus $A_i^c \in \mathcal{R}$ (this follows from property 2). Since by mathematical induction one can show that property 2 holds for every finite number of sets in $\mathcal{R}$, we can conclude that $R_1^c = \bigcap_{i=1}^{n} A_i^c \quad \in \mathcal{R}$.

Easy 4 (Test Feb-05), *1.16 (Notes Feb-01)* ✓

Problem: Let $\mathcal{C} = (C_1, C_2, \ldots, C_m)$ be a finite partition of Ω. Show that

$$\mathcal{R} := \Big\{ \bigcup_{i \in \alpha} C_i : \alpha \subseteq (1, \ldots, m) \Big\}$$

is a field on Ω and that it is the smallest field containing $\mathcal{C}$.

$\mathcal{C} = (C_1, \ldots, C_m)$ finite partition of Ω

is a generating partition of $\mathcal{R}$; $\mathcal{R} := \big\{ \bigcup_{i \in \alpha} C_i : \alpha \subseteq (1, \ldots, m) \big\}$

(a) Show that $\mathcal{R}$ is a field on Ω:

1. by def. of partition: $\bigcup_{i=1}^m = \Omega \quad \Rightarrow \quad \Omega \in \mathcal{R}$

   $\phi =$ union of nothing $\quad \Rightarrow \quad \phi \in \mathcal{R}$

2. take $R_1 = \bigcup_{i \in \alpha_1} C_i$ and $R_2 = \bigcup_{j \in \alpha_2} C_j$, $quad \alpha_1, \alpha_2 \subseteq (1, \ldots, m)$. Then $\alpha_1 \cup \alpha_2 \subseteq (1, \ldots, m)$ and $\alpha_1 \cap \alpha_2 \subseteq (1, \ldots, m)$

   $R_1 \cup R_2 = \bigcup_{k \in (\alpha_1 \cup \alpha_2)} C_k \ \in \mathcal{R}$

   $R_1 \cap R_2 = \bigcap_{l \in (\alpha_1 \cap \alpha_2)} C_l \ \in \mathcal{R}$

3. take $R_1 = \bigcup_{i \in \alpha_1} C_i$. Then $R_1^c = \bigcup_{i \in \alpha_1^c} C_i \ \in \mathcal{R}$ as $\alpha_1 \subseteq (1, \ldots, m) \ \Rightarrow \ \alpha_1^c \subseteq (1, \ldots, m)$

(b) Show that $\mathcal{R}$ is the smallest field containing $\mathcal{C}$: Let $\mathcal{F}$ be any field s.t. $\mathcal{F} \supseteq \mathcal{C}$. Then $\mathcal{F}$ contains unions of elements in $\mathcal{C} \Rightarrow \mathcal{F} \supseteq \mathcal{R} \quad \forall \mathcal{F} \supseteq \mathcal{C}$!

Easy 5 (Test Feb-05), *1.18 (Notes Feb-01)* ✓

Problem: Let $\mathcal{R}$ be a finite field and let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be the generating partition. Show that for every choice of numbers $a_i \geq 0$ there exists exactly one content $\mu|\mathcal{R}$ such that $\mu(C_i) = a_i$.

Correct solution by Prof. Strasser:

If there exists a content $\mu|\mathcal{R}$ such that $\mu(C_i) = a_i$ then by additivity of a content it must be true that

$$D = \bigcup_{j\in\alpha} C_j \text{ implies } \mu(D) = \sum_{j\in\alpha} \mu(C_j) = \sum_{j\in\alpha} a_j$$

This shows that there is at most one content with the desired property. But we have also to show there is a content at all ! Therefore we have to take this equation as a preliminary definition and then try to show that this definition satisfies the properties of a content.

It is certailny nonnegative and it is obvious that $\mu \geq 0$. So we need only show that it is additive. For this let D_1 and D_2 be disjoint sets in $\mathcal{R}$ such that

$$D_1 = \bigcup_{j\in\alpha_1} C_j \text{ and } D_2 = \bigcup_{j\in\alpha_2} C_j$$

where in view of disjointness we have $\alpha_1 \cap \alpha_2 = \emptyset$. Then

$$\mu(D_1 \cup D_2) = \mu(\bigcup_{j\in\alpha_1\cup\alpha_2} C_j) = \sum_{j\in\alpha_1\cup\alpha_2} a_j = \sum_{j\in\alpha_1} a_j + \sum_{j\in\alpha_2} a_j = \mu(D_1) + \mu(D_2).$$

Easy 6a (Test Feb-05), *1.23 (Notes Feb-01)* ✓

PROBLEM: (a) Let $(\Omega, \mathcal{F})$ be any measurable space. Let $x \in \Omega$ some point and keep it fixed. For every $A \in \mathcal{F}$ define

$$\delta_x(A) = \begin{cases} 1 & \text{whenever } x \in A \\ 0 & \text{whenever } x \notin A \end{cases}$$

Show that $\delta_x : A \mapsto \delta_x(A)$ is a measure (the *one-point measure* at the point x).

(b) Show that every finite linear combination of measures with nonnegative coefficients is a measure.

——

We already know that :(Ω, F) is a σ - fields therefore we need just to show that δ is a σ - additive content.

One can simply check the condition of a σ -additive content.

Having Taken any sequence of disjoint A_is it is obvious that (x belongs just to one A_i) OR (X does not belong to any A_i). so we have two cases:

1) if x $\in A_k$ then x$\notin A_j$, j$\neq$ k then $\mu(\bigcup A_i) = 1$ which is equal to $\mu(A_k) + \sum_{j\neq k} \mu(A_j)$= $\sum \mu(A_i)$

2) if $\forall i$ x $\notin A_i$ then it obvious that x $\notin \bigcup(A_i)$ therefore $\mu(\bigcup A_i) = 0$. On the other hand $\mu(A_i) = 0$ for all i. $\Rightarrow \mu(\bigcup A_i) = \sum \mu(A_i)$

Easy 6b (Test Feb-05), *1.24a (Notes Feb-01)* ✗

Prof. Strasser: *basically OK. discuss also the remaining properties of a measure, not only additivity.*

Problem:

——

First we prove it for the case of two measure. Take μ_1 and μ_2 as two measures on the same σ field. Assume that μ_3 is a linear combination of them i.e $\mu_3 = \alpha_1\mu_1 + \alpha_2\mu_2$.

$\mu_3(\bigcup A_i) = \alpha_1\mu_1(\bigcup A_i) + \alpha_2\mu_2(\bigcup A_i) = \alpha_1 \sum \mu_1(A_i) + \alpha_2 \sum \mu_2(A_i) = \sum \alpha_1\mu_1(A_i)$ $+ \sum \alpha 2\mu_2(A_i) = \sum(\alpha_1\mu_1(A_i) + \alpha_1\mu_1(A_i)) = \sum \mu_3(A_i)$

The case of more than two measures is a natural extension of the previous proof. We already know that it is true for two measures. Also if it is true for n measures then for the case of n+1 measures, the problem is equal to the sum of first n measures and the measure $(n+1)^{th}$. The sum of the first n measure is a measure itself so the problem is converted to the case of sum of two measures which we did it. So it has been proved by induction.

Easy 7 (Test Feb-05), *1.30 (Notes Feb-01)* ✗

Prof. Strasser: *is not OK: the solution on the blackbord in the exercise lesson was OK I suppose there should be a correct handwritten version.*

PROBLEM: Show that the σ-field on $\mathbb{N}$ which is generated by the one-point sets of $\mathbb{N}$ is $\mathcal{F} = 2^{\mathbb{N}}$.

——

Since N is a countable set we can use the idea similar to the problem no 1.16 . it is obvious that the collection of all Natural numbers , called S here, is a partition of N consisting of one-point sets.

S= $\{C_i : C_i = i, i \in N\}$

On the other hand and according to the 1.16 (supposed to has been solved before) $2^N = \{\bigcup_{i \in N} C_i\}$ is the smallest field containing S. also 2^N is a σ field. Therefore 2^N is generated by N.

Easy 8 (Test Feb-05), *2.3 (Notes Feb-01)* ✗

Prof. Strasser: *This is not an answer but only a hint.*

PROBLEM: Let $f : \Omega \to \mathbb{R}$ be measurable. Show that μ^f is a measure on $\mathcal{B}$.

——

To prove that μ^f is a measure we should show that it is a σ additive content.

$\mu^f(\bigcup A_i) = \mu(f \in \bigcup A_i)$. On the other hand we know that if all A_is are disjoint then f belongs to at most one of them. Similar to what we have shown in exercise 1.23 (E6a) one can easily show that μ^f is a σ additive content.

Easy 9 (Test Feb-05), *2.4 (Notes Feb-01)* ✗

Prof. Strasser: *(a) is OK but too long, (b) is missing*

PROBLEM: Let $\Omega, \mathcal{F}, \mu)$ be a measure space and let $f = 1_A$ where $A \subseteq \Omega$.

(a) Show that f is $\mathcal{F}$-measurable iff $A \in \mathcal{F}$.

(b) Find μ^f.

(Ω, F, μ) - measure space, f= 1_A= $\begin{cases} 1, & \omega \in A, A \subseteq \Omega \\ 0, & \omega \notin A \end{cases}$

a) f is F-meas. iff A $\in$ F

(i) Let f be F-meas. Suppose A $\notin$ F

Since f is F-meas. we we have $\forall B \in \beta$ (f $\in$ $B) = f^{-1}(B) \in$ F

But $f^{-1}(B) = \begin{cases} \Omega, & 0, 1 \in \beta \\ A^c, & 0 \in \beta, 1 \notin \beta \\ A, & 0 \notin \beta, 1 \in \beta \\ \phi, & 0, 1 \notin \beta \end{cases}$

which contradicts the assumption that A $\notin$ F.

(ii) Now let A $\in$ F. We need to show $f^{-1}(B) \in$ F. Since

f= 1_A= $\begin{cases} 1, & \omega \in A, A \subseteq \Omega \\ 0, & \omega \notin A \end{cases}$

we had $f^{-1}(B) = \begin{cases} \Omega, & 0, 1 \in \beta \\ A^c, & 0 \in \beta, 1 \notin \beta \\ A, & 0 \notin \beta, 1 \in \beta \\ \phi, & 0, 1 \notin \beta \end{cases}$

Since F is a σ-field, $\Omega \in$ F and $\Omega^c = \phi \in$ F. Since we assumed A$\in$F , $A^c \in$ F. Hence $f^{-1}(B) \in F$ and f=F-meas.

Easy 10 (Test Feb-05), *2.5 (Notes Feb-01)* ✗

Prof. Strasser: *(a) is OK, but you should state that you are using the canonical representation and explain what this is. (b) is wrong. the answer is* $\mu^f(B) = \mu(f \in B) = \sum_{i:a_i \in B} \mu(F_i) = \sum \mu(F_i)\delta_{a_i}(B)$

Problem: Let $\Omega, \mathcal{F}, \mu)$ be a measure space and let $f : \Omega \to \mathbb{R}$ be a simple function.

(a) Show that f is $\mathcal{F}$-measurable iff all sets of the canonical representation are in $\mathcal{F}$.

(b) Find μ^f.

---

a f is $\mathcal{F}$-measurable iff $F_i \in \mathcal{F}\,\forall i$; $f = \sum_{i=1}^n a_i 1_{F_i}$.

- $\Rightarrow$ Let f be $\mathcal{F}$-measurable and suppose that for some j $F_j \notin \mathcal{F}$. But since $\{a_1, \ldots, a_n\}$ are distinct values and f is $\mathcal{F}$-measurable, $f^{-1}(a_i) \in \mathcal{F}\,\forall i$. But then $f^{-1}(a_j) \in \mathcal{F}$ by $\mathcal{F}$-measurability of f. But this contradicts $F_j \notin \mathcal{F}$.

- $\Leftarrow$ Now let $F_i \in \mathcal{F}\ \forall i = 1, \ldots, n$. If f is $\mathcal{F}$-measurable, then we must have $f^{-1}(B) \in \mathcal{F}$. Since $f = \sum_{i=1}^n a_i 1_{F_i}$ with $a_i \neq a_j\,\forall i \neq j$, we must have $f^{-1}(a_i) \in \mathcal{F}\ \forall i$. But this follows easily since $(f^{-1}(a_i)) = F_i \in \mathcal{F}$.

  Hence, f is $\mathcal{F}$-measurable.

b $\mu^f(B) = \mu(f \in B) = \mu(f^{-1}(B))$

Let $f_i := a_i 1_{F_i}$, then $f = \sum_{i=1}^n a_i 1_{F_i} = \sum_{i=1}^n f_i$. By definition $1_{F_i} = 1$ for $w \in F_i$ and 0 otherwise.

Then for each f_i we have:

$$f_i^{-1}(B) = \begin{cases} F_i & a_i \in B \quad \forall i = 1, \ldots, n \\ F_i^c & a_i \notin B \end{cases}$$

and

$$\mu\left(f_i^{-1}(B)\right) = \begin{cases} \mu(F_i) & a_i \in B \\ \mu(F_i^c) & a_i \notin B \end{cases}$$

Then $\mu^f(B) = \sum_{i=1}^n \mu\left(f_i^{-1}(B)\right)$.

Easy 11 (Test Feb-05), *2.22 (Notes Feb-01)* ✗

Prof. Strasser: *(a) OK. It would sufficient to observe that F(x+1/n)-¿F(x). (b) Equality on the intervals is not euqlity of measures. Use the measure extension theorem to shoe equality of measures.*

Problem: (a) Show that any distribution function is right-continuous.
(b) Show that the distribution $P^X = \lambda_F$.

Let X be a random variable. Then the function $F_X : \Re \mapsto [0, \infty]$ defined by

$$F_X(x) := P(X \leq x)\ , x \in \Re$$

is the distribution function of X. The distribution of X is P^X, i.e. the image of P under X defined by

$$P^X(B) := P(X^{-1}(B)) = P(X \in B),\ B \in \mathcal{B}$$

or equivalently

$$P^X(B) := P(X^{-1}(B)) = P(\{\omega \in \Omega : X(\omega) \in B\}).$$

Thus the distribution function F_X determines the values of the distribution P^Xon intervals by

$$P^X((a, b]) = F(b) - F(a).$$

(a) Show that any distribution function is right-continuous.

Fix $x \in \Re$ and for $n \in \aleph$ set $B_n = \{\omega : X(\omega) \leq x + \frac{1}{n}\}$. Then $B_1 \supseteq B_2 \ldots$ and $\bigcap_n B_n = \{\omega : X(\omega) \leq x\}$. It follows that by Lemma 1.7

$$P(B_n) \to P(\{\omega : X(\omega) \leq x\}) = F_X(x)$$

Let $\epsilon > 0$ be given and let n_0 be such that $|P(B_{n0}) - F_X(x)| < \epsilon$. Then we have

$$0 \leq P(B_{n0}) - F_X(x) < \epsilon$$

which means that $0 \leq F_X(x + \frac{1}{n0}) - F_X(x) < \epsilon$. Let $0 < h < \frac{1}{n0}$. Then we get

$$\begin{aligned}
0 &\leq F_X(x+h) - F_X(x) \\
&\leq F_X(x + \frac{1}{n0}) - F_X(x) \\
&< \epsilon
\end{aligned}$$

(b) Show that the distribution $P^X = \lambda_F$*.* Let λ_F denote a right-continuous (left open, right-closed) increasing function. Then $\lambda_F((a,b]) = F(b) - F(a)$ From above (and section 2.4) $P^X((a,b]) = F(b) - F(a) = \lambda_F((a,b])$. Which implies that $P^X = \lambda_F$

Easy 12 (Test Feb-05), *3.6 (Notes Feb-01)* ✓

> Problem: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $g \in \mathcal{L}(\mathcal{F})$. Then for every $f \in \mathcal{S}_+(\mathcal{B})$
> $$\int f \circ g \, d\mu = \int f \, d\mu^g$$

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $g \in \mathcal{L}(\mathcal{F})$ Then for every $f \in \mathcal{S}^+(\mathcal{B})$

$$\int f \circ g \, d\mu = \int f \, d\mu^g$$

Proof: As $f \in \mathcal{S}^+(\mathcal{B})$ we can define

$$f = \sum_{i=1}^{n} \alpha_i 1_{B_i}$$

Then

$$f \circ g(\omega) = f(g(\omega)) = \sum_{i=1}^{n} \alpha_i 1_{g(\omega)\in B_i} = \sum_{i=1}^{n} \alpha_i 1_{g^{-1}(B_i)}$$

According to definition 3.1

$$\int f \circ g d\mu = \sum_{i=1}^{n} \alpha_i \mu(g^{-1}(B_i)) = \sum_{i=1}^{n} \alpha_i \mu^g(B_i) = \int f d\mu^g$$

Easy 13 (Test Feb-05), *3.7 (Notes Feb-01)* ✗

Prof. Strasser: *Omit $\int f dF$ and all is OK*

PROBLEM: Let $(\Omega, \mathcal{F}, P)$ be a probability space and X a random variable with distribution function F. Explain the formula

$$E(f \circ X) = \int f \, d\lambda_F$$

Let $(\Omega, \mathcal{F}, P)$ a probability space and X a random variable. Explain the formula

$$\int_A E(f \circ X) = \int f \, d\lambda_F$$

Proof: According to the definition of expectation we have

$$E(f \circ X) = \int f \circ X dP$$

From the transformation theorem 3.5 we can rewrite this equation into

$$\int f \circ X dP = \int f dP^X$$

where P^X is the probability distribution of X and can be substituted with the distribution function F

$$\int f dP^X = \int f dF$$

Recall the Lebesgue-Stieltjes content $\lambda_F|\Re$, which is the usual way to define probability distribution by distribution functions in probability theory, so we obtain

$$\int f dF = \int f d\lambda_F$$

Combining the equations above we finally end up with

$$E(f \circ X) = \int f d\lambda_F$$

Easy 14 (Test Feb-05), *3.18 (Notes Feb-01)* ✓

PROBLEM: Show that $f \in \mathcal{L}(\mathcal{F})$ is μ-integrable iff $\int |f| \, d\mu < \infty$.

Let $f \in \mathcal{L}(\mathcal{F})$ be μ-integrable. Then by definition f^+ and f^- are μ-integrable, hence we get for the integral of $|f| = f^+ + f^-$

$$\int |f| d\mu = \int (f^+ + f^-) d\mu = \int f^+ d\mu + \int f^- d\mu < \infty \, .$$

Now suppose $\int |f| d\mu < \infty$. Then

$$\infty > \int |f| d\mu = \int (f^+ + f^-) d\mu = \int f^+ d\mu + \int f^- d\mu$$

and

$$\int f^+ d\mu < \infty \qquad \text{and} \qquad \int f^- d\mu < \infty \, .$$

Thus (by definition) f is μ-integrable.

Easy 15 (Test Feb-05), *3.24 (Notes Feb-01)* ✓

PROBLEM: Let f be a measurable function and assume that there is an integrable function g such that $|f| \leq g$. Then f is integrable.

——

By the isotonic property we know $\int |f| d\mu \leq \int g d\mu$. But g is integrable, hence f is integrable too: $\int |f| d\mu \leq \int g d\mu < \infty$.

Easy 16 (Test Feb-05), *4.5 (Notes Feb-01)* ✓

PROBLEM: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f \in \mathcal{L}^+(\mathcal{F})$. Show that $\nu : A \mapsto \int_A f\, d\mu$, $A \in \mathcal{F}$ is a measure.

——

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f \in \mathcal{L}^+(\mathcal{F})$. Show that $\nu : A \mapsto \int_A f\, d\mu, A \in \mathcal{F}$ is a measure. Proof:(i)

$$\nu(A) = \int_A f d\mu = \int 1_A f d\mu \geq 0 \; \forall A \in \mathcal{F}$$

$$\Rightarrow \nu(A) \in [0, \infty] \;\; \forall A \in \mathcal{F}$$

(ii)

$$\nu(\phi) \int_\phi f d\mu = \int 1_\phi f d\mu = \int 0 * f d\mu = \int 0 * d\mu = 0.$$

Since

$$1_\phi(\omega) = 0 \; \forall \omega \in \Omega$$

(iii) Let $A_i \in \mathcal{F}$ be disjoint, then

$$\nu(\bigcup_{i=1}^{\infty} A_i) = \int_{\bigcup\limits_{i=1}^{\infty} A_i} f d\mu = \int 1_{\bigcup\limits_{i=1}^{\infty} A_i} f d\mu = \int_{\sum\limits_{i=1}^{\infty} 1_{A_i}} f d\mu = \int (\sum_{i=1}^{\infty} 1_{A_i} f) d\mu$$

$$= {}^1 \sum_{i=1}^{\infty} \int 1_{A_i} f d\mu = \sum_{i=1}^{\infty} \int_{A_i} f d\mu = \sum_{i=1}^{\infty} \nu(A_i)$$

Ad footnote 1:

$$f \in \mathcal{L}^+(\mathcal{F}) \Rightarrow 1_{A_k} f \in \mathcal{L}^+(\mathcal{F})$$

[1] 1

Which implies that

$$\sum_{k=1}^{n} 1_{A_k} f \in \mathcal{L}^+(\mathcal{F}).$$

Also $\sum_{k=1}^{n} 1_{A_k} f \uparrow f$ as $n \to \infty$. Which implies that

$$\sum_{k=1}^{\infty} \int 1_{A_k} f d\mu = \lim_{n\to\infty} \sum_{k=1}^{n} \int 1_{A_k} f d\mu$$

$$= \lim_{n\to\infty} \int \sum_{k=1}^{n} 1_{A_k} f d\mu \overbrace{=}^{\text{Beppo Levi}} \lim_{n\to\infty} \sum_{k=1}^{n} 1_{A_k} f d\mu = \int \sum_{k=1}^{\infty} 1_{A_k} f d\mu$$

Easy 17 (Test Feb-05), *4.9 (Notes Feb-01)* ✓

Problem: Let $\nu = f\mu$. Show that $\mu(A) = 0$ implies $\nu(A) = 0$, $A \in \mathcal{F}$.

Let $\nu = f\mu$. Show that $\mu(A) = 0$ implies $\nu(A) = 0$, $A \in \mathcal{F}$.

Proof:

$$\nu : A \mapsto \int_A f d\mu,\ A \in \mathcal{F}$$

$$\nu = f\mu \text{ and } f := \frac{d\nu}{d\mu}$$

We know from Bauer (German version page 81) that

$$\int f d\mu = 0 \Leftrightarrow\ f = 0\mu - a.e.$$

(This lemma is similar to our definition on page 22 (latest version script)) From Bauer we see that this lemma can be rewritten in the following way for some set $N := \{f \neq 0\} = \{f > 0\}$

$$\int_N f d\mu = 0 \Leftrightarrow \mu(N) = 0$$

Applying this for set A gives obviously

$$\mu(A) = 0 \Rightarrow \int_A f d\mu = 0$$

As $\nu(A) = \int_A f d\mu$ it follows that

$$\nu(A) = 0.$$

as we were supposed to show.

Easy 18 (Test Feb-05), *4.14 (Notes Feb-01)* ✓

PROBLEM: Let $\nu = f\mu$. Discuss the validity of

$$\int f\, d\nu = \int f \frac{d\nu}{d\mu} d\mu$$

——

correct version of question and solution by Prof. Strasser

Let $\nu = \dfrac{d\nu}{d\mu}\mu$.Discuss the validity of

$$\int f\, d\nu = \int f \frac{d\nu}{d\mu} d\mu.$$

Hint: Prove it for $f \in \mathcal{S}^+(\mathcal{F})$ and extend it by measure theoretic induction.

Proof: Let $f = \sum_{i=1}^n \alpha_i 1_{A_i}$ Then $\int f\, d\nu = \sum_{i=1}^n \alpha_i \nu(A_i)$ Because of

$$\nu(A_i) = \int_{A_i} \frac{d\nu}{d\mu} d\mu$$

We get

$$\begin{aligned}
\int f\, d\nu &= \sum_{i=1}^n \alpha_i \int_{A_i} \frac{d\nu}{d\mu} d\mu \\
&= \sum_{i=1}^n \int \alpha_i 1_{A_i} \frac{d\nu}{d\mu} d\mu \\
&= \int (\sum_{i=1}^n 1_{A_i}\alpha_i) \frac{d\nu}{d\mu} d\mu \\
&= \int f \frac{d\nu}{d\mu} d\mu
\end{aligned}$$

Now by standard measure theoretic induction we get the validity for nonnegative measurable functions (use Levi's theorem) and then for integrable function by applying the equation to the positive and the negative parts (write down details !)

Easy 19 (Test Feb-05), *5.2a (Notes Feb-01)* ✓

PROBLEM: Let X be an indicator random variable. Find $\sigma(X)$.

$X = 1_A$ for some $A \in \mathcal{F}$ (note that $1_A \in \{0,1\}$). $1_A^{-1}(0) = A^C$, $1_A^{-1}(1) = A$

Let $B \in \mathcal{B}(R)$. Then

$$X^{-1}(B) = (X \in B) = (1_A \in B) = \begin{cases} \phi & 0 \notin B, 1 \notin B \\ A & 0 \notin B, 1 \in B \\ A^C & 0 \in B, 1 \notin B \\ \Omega & 0 \in B, 1 \in B \end{cases}$$

$$\Rightarrow \{(X \in B) | B \in \mathcal{B}(R)\} = \{\phi, A, A^C, \Omega\} =: \mathcal{C}$$

Note that $\mathcal{C}$ is a σ-field according to the definitions 1.1 and 1.20 (as of Notes 5/2) of the script.

$$\Rightarrow \sigma(X) = \sigma((X \in B), B \in \mathcal{B}(R)\} = \sigma(\mathcal{C}) = \mathcal{C}$$

since $\sigma(X)$ is the smallest σ-field containing $\mathcal{C}$ (which is a σ-field too!); so $\sigma(X) = \{\phi, A, A^C, \Omega\}$

Easy 20 (Test Feb-05), *5.2b (Notes Feb-01)* ✗

Prof. Strasser: *is OK (with misprints) but complicated. why don't you say it in this way: all inverse images of X are unions of sets in the partition. we know (from where ?) that these unions constitute a field.*

PROBLEM: Let X be a simple random variable. Find $\sigma(X)$.

$X \in \mathcal{S}(\mathcal{F})$ (= simple measurable function) $\Rightarrow X(\Omega) = \{a_1, \ldots, a_n\} \subset R$

$\Rightarrow X = \sum_{i=1}^n a_i 1_{A_i}$, where $A_i := X^{-1}(a_i)$ is a partition of Ω.

Lemma: $\{(X \in B) | B \in \mathcal{B}(R)\} = \{\bigcup_{i \in \mathcal{I}} A_i | \mathcal{I} \subset \{1, \ldots\}\} =: \mathcal{C}$

proof of "$\subset$": Let $B \in \mathcal{B}(R)$, $\mathcal{I} := \{i \in N | a_i \in B\}$

$\Rightarrow (X \in B) = (\sum_{i=1}^n 1_{A_i} \in B) = (\sum_{i=1}^n 1_{A_i})^{-1}(B) = \bigcup_{i \in \mathcal{I}} A_i$

proof of "$\supset$": Let $\mathcal{I} \subset \{1, \ldots, n\}$, $B := \{a_i | i \in \mathcal{I}\} \in \mathcal{B}(R)$ since B is countable.

$\Rightarrow \bigcup_{i \in \mathcal{I}} A_i = (\sum_{i=1}^n a_i 1_{A_i})^{-1}(B) = (\sum_{i=1}^n 1_{A_i} \in B) = (X \in B)$

Using this lemma, we have:

$$\begin{aligned} \sigma(X) &= \sigma((X \in B), B \in \mathcal{B}(R)) \\ &= \sigma\left(\bigcup_{i \in \mathcal{I}} A_i | \mathcal{I} \subset \{1, \ldots, n\}\right) \\ &= \sigma(\mathcal{C}) = \mathcal{C} \end{aligned}$$

since $\mathcal{C}$ itself is a σ-field according to the lecture notes (as of 5/2; combining problem 1.16 and definition 1.20).

$$\text{so } \sigma(X) = \left\{\bigcup_{i \in \mathcal{I}} A_i | \mathcal{I} \subset \{1, \ldots, n\}\right\} = \sigma(A_i, \ldots, A_n).$$

Easy 21 (Test Feb-05), *6.3 (Notes Feb-01)* ✗

Prof. Strasser: *That is nonsense. I gave the correct proof at the blackboard.*

PROBLEM: Find the conditional expectation given a finite field.

——

We know that every finite field is generated by a partition (here: $\bigcup_{i=1}^m C_i$). So let $\mathcal{A} = \sigma(C_1, \ldots, C_m)$.

Then the conditional expectation of X given $\mathcal{A}$ is the following:

$$E(X/\mathcal{A}) = \sum_{i=1}^m E(X/C_i) 1_{C_i},$$

with
$$E(X/C_i) = \frac{1}{P(C_i)} \int_{C_i} XdP.$$
proof: $1_A = \sum_{i=1}^m 1_{C_i}$, so we can write
$$\begin{aligned}
E(X/\mathcal{A}) &= \int_{\mathcal{A}} XdP = \int 1_{\mathcal{A}} XdP = \sum_{i=1}^{m} \int 1_{C_i} XdP \\
&= \sum_{i=1}^{m} \int_{C_i} XdP = \sum_{i=1}^{m} \frac{P(C_i)}{P(C_i)} \int_{C_i} XdP \\
&= \sum_{i=1}^{m} P(C_i) E(X/C_i) = \sum_{i=1}^{m} E(X/C_i) 1_{C_i}
\end{aligned}$$

Easy 22 (Test Feb-05), *6.4a (Notes Feb-01)* ✓

PROBLEM: Show that $E(E(X|\mathcal{A})) = E(X)$.

——

Let $Y = E(X \backslash \mathcal{A})$. Then
$$\begin{aligned}
E[E(X \backslash \mathcal{A})] &= E(Y) \\
&= \int_{\Omega} Y dp \qquad \text{(by definition of expectation)} \\
&= \int_{\Omega} X dp \qquad \text{(by definition of conditional expectation)} \\
&= E(X) \qquad \text{(by definition of expectation)}
\end{aligned}$$

Easy 23 (Test Feb-05), *6.11 (Notes Feb-01)* ✓

PROBLEM: Let X and Y be square integrable. If X is $\mathcal{A}$-measurable and Y is independent of $\mathcal{A}$ then
$$E(XY|\mathcal{A}) = XE(Y).$$

——

From the Redundant Conditioning Theorem 6.9 (lecture notes from 2006-02-01) we know that if X and Y are square-integrable and X is $\mathcal{A}$-measurable, then

$$E(XY|\mathcal{A}) = XE(Y|\mathcal{A})$$

Replicating the proof of exercise 6.7c (notes from 2006-02-01), we will now show that $E(Y|\mathcal{A}) = E(Y)$:

If Y is independent of $A \ \forall A \in \mathcal{A}$, then Y is also independent of $1_{\mathcal{A}}$

$$\begin{aligned}\int_A E(Y|A)dP = \int_A YdP = \int 1_A YdP \\ = \int E(1_A)E(Y)dP = \int_A E(Y)dP \\ \Rightarrow E(Y|A) = E(Y) \text{ P.-a.e. } \ \forall A \in \mathcal{A}\end{aligned}$$

Therefore $XE(Y|\mathcal{A}) = XE(Y)$
and so $E(XY|\mathcal{A}) = XE(Y)$

Easy 24 (Test Feb-05), *7.13 (Notes Feb-01)* ✗

Prof. Strasser: *is nonsense. The solution was presented correctly on the blackboard. There should be a correct handwritten solution.*

Problem: Let $(X_n)_{n\geq 0}$ be adapted. Show that the *hitting time* or *first passage time*

$$\tau = \min\{k \geq 0 : X_k \in B\}$$

is a stopping time for any $B \in \mathcal{B}$.

——

Let Xn be Fn-measurable; $\forall$ n and $\tau = min\{k \geq 0 : Xk \in B\}$ $\forall B \in \mathcal{B}$ Show that $(\tau = k) \in F_k \forall B \in \mathcal{B}$ By definition:

$$(\tau = k) = \{X_1 \notin B\}, \{X_2\} \notin B \ldots \{X_{k-1}\}, \{X_k \in B\} \tag{1}$$

$$(\tau = k) = \{X_1 \notin B\} \cap \{X_2 \notin B\} \cap \ldots \cap \{X_k \in B\} \tag{2}$$

$$(\tau = k) = \{X_1\}^c \cap \{X_2\}^c \cap \ldots \cap \{X_k \in B\} \in \sigma(X_1, \ldots X_k) \tag{3}$$

Denote the set $X = (X_1 \in B, \ \ldots X_k \in B\})$ Since X_i are F_i-measureable, we have $X^{-1}B \in F_k$ But since $(\tau = k) = X$ and from (1),(2) and (3) $X^{-1}B \in F_k$ implies that $(\tau = k) \in F_k$. Hence,$(\tau = k)$ defines a stopping time.

Easy 25 (Test Feb-05), *7.14 (Notes Feb-01)* ✓

PROBLEM: Let (X_k) be a sequence adapted to $(\mathcal{F}_k)$ and let τ be a finite stopping time. Then X_τ is a random variable.

Let (X_k) be a sequence adapted to $(\mathcal{F}_k)$ and let τ be a finite stopping time. Then X_τ is a random variable.

Let $(X_k) = (X_1, X_2, \ldots)$ for $k = 1, 2, \ldots$ and let $\mathcal{F}_k = \sigma(X_1, X_2, \ldots)$ be the past of the sequence.

(X_k) is adapted to $(F_k)_{k \geq 0}$ if (X_k) is $(\mathcal{F}_k)$-measurable for all k. $\tau : \Omega \mapsto N_0 \cup \{\infty\}$ is a stopping time relative to the filtration if $(\tau = k) \in \mathcal{F}_k$ for all $k \in N$.

Using the causality theorem we get $1_{\tau=k} = f_k(X_1, X_2, \ldots X_k)$

$$X_\tau = X_k \text{ for } (\tau = k)$$

$$X_\tau = \sum_{k=0}^{\infty} X_k 1_{(\tau=k)} \text{ with } X_k 1_{(\tau=k)} \text{ being a r.v. and thus } \sum_{k=0}^{\infty} X_k 1_{\tau=k} \text{ being a r.v.}$$

Easy 26 (Test Feb-05), *7.19 (Notes Feb-01)* ✓

PROBLEM: Let $(S_n)_{n\geq 0}$ be a random walk (starting at $S_0 = 0$) with discrete steps $+1$ and -1. Let $\tau := \min(k \geq 0 : S_k = -a \text{ or } S_k = b)$.

(a) Calculate $E(S_\tau)$.

(b) Use Wald's equation $E(S_\tau) = \mu E(\tau)$ to obtain $E(\tau)$ for a non-symmetric random walk.

(c) Use equation $E(S_\tau^2) = E(\tau)$ to find $E(\tau)$ for any random walk.

——

a Let τ_{-a} and τ_b denote the hitting times of the one-sided boundaries $-a$ and b, so $(S_\tau = -a) = (\tau_{-a} < \tau_b)$ and $(S_\tau = b) = (\tau_{-a} > \tau_b)$

By discussion 7.2 (as of 5/2) we receive (noting that $S_0 = 0$) and setting initial wealth V_0 to a, we get:

$$\begin{aligned} P(S_\tau = -a) &= P(\tau_{-a} < \tau_b) = P(\tau_0 < \tau_{a+b} | V_0 = a) = q_0(a) \\ P(S_\tau = b) &= P(\tau_b < \tau_{-a}) = P(\tau_{a+b} < \tau_0 | V_0 = a) = q_{a+b}(a) \end{aligned}$$

$$q_0(a) \rightarrow \begin{cases} \frac{\left(\frac{p}{1-p}\right)^b - 1}{\left(\frac{p}{1-p}\right)^{a+b} - 1} & p \neq 1/2 \\ \frac{b}{a+b} & p = 1/2 \end{cases}$$

and

$$q_{a+b}(a) \rightarrow \begin{cases} \frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^{a+b} - 1} & p \neq 1/2 \\ \frac{a}{a+b} & p = 1/2 \end{cases}$$

We also know that $E(S_\tau) = bP(S_\tau = b) - aP(S_\tau = -a) = bq_{a+b}(a) - aq_0(a)$.

Using this we receive for $p \neq 1/2$:

$$E(S_\tau) = b\frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^{a+b} - 1} - a\frac{\left(\frac{p}{1-p}\right)^b - 1}{\left(\frac{p}{1-p}\right)^{a+b} - 1}$$

and for $p = 1/2$:

$$E(S_\tau) = b\frac{a}{a+b} - a\frac{b}{a+b} = 0$$

b Wald's Equation: $E(S_\tau) = \mu E(\tau)$

By discussion 7.17 (as of 5/2) we can use the approximation $E(S_{\tau\cap n} = \mu E(\tau \cap n)$, which converges by Lebesgue's dominated convergence theorem and Beppo Levi's Theorem, and hence Wald's Equation is valid in our case. So for $p \neq 1/2$ we get:

$$E(\tau) = \frac{E(S_\tau)}{\mu} = \left[b \frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^{a+b} - 1} - a \frac{\left(\frac{p}{1-p}\right)^b - 1}{\left(\frac{p}{1-p}\right)^{a+b} - 1} \right] \frac{1}{2p-1}.$$

c By a), we know that $E(S_\tau) = bP(S_\tau = b) - aP(S_\tau = -a)$. So

$$\begin{aligned} E(S_\tau^2) &= b^2 P(S_\tau = b) - a^2 P(S_\tau = -a) \\ &= b^2 q_{a+b}(a) - a^2 q_0(a) \end{aligned}$$

For $p \neq 1/2$, we receive

$$E(S_\tau^2) = b^2 \frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^{a+b} - 1} - a^2 \frac{\left(\frac{p}{1-p}\right)^b - 1}{\left(\frac{p}{1-p}\right)^{a+b} - 1} = E(\tau)$$

For $p = 1/2$, we get

$$E(S_\tau^2) = b^2 \frac{a}{a+b} - a^2 \frac{b}{a+b} = \frac{ab(b-a)}{a+b} = E(\tau)$$

Easy 27 (Test Feb-05), *7.20 (Notes Feb-01)* ✓

PROBLEM: Let (S_k) be a symmetric random walk and let $\tau := \min(k \geq 0 : S_k = 1)$. Show that $E(\tau) = \infty$.

Prof. Strasser: *OK, tell us what is* $E(S_\tau)$

Let (S_k) be a symmetric random walk and let $\tau := \min(k \geq 0 : S_k = 1)$. Show that $E(\tau) = \infty$.

Assume that $E(\tau) < \infty$, so τ is a bounded stopping time. Hence, we may apply Wald's equation

$$E(S_\tau) = \mu E(\tau)$$

So we get $E(\tau) = \frac{E(S_\tau}{\mu}$ But $\mu = 0.5 * (1) + 0.5 * (-1) = 0$, so $E(\tau) = \infty$ Which is a contradiction.

Easy 28 (Test Feb-05), *7.35 (Notes Feb-01)* ✗

Prof. Strasser: *this OK but the complicated way. if you use the definition of a martingale in terms of cond exp things are much easier and very similar to the martingale proof of the wiener process*

Problem: Let $S_n = X_1 + X_2 + \cdots + X_n$ where (X_i) are independent identically distributed (i.i.d.) and integrable random variables with $E(X_i) = \mu$.
(a) Show that (S_n) is a martingale iff $\mu = 0$.
(b) Which kind of martingale property hold if $\mu \neq 0$?

---

a) Show that (S_n) is a martingale iff $\mu = 0$.

Let σ and τ be discrete and bounded stopping times (with $\sigma \leq \tau \leq K$).

From the proof of Theorem 7.26 (Notes 5/2) (noting that H_i is 1 for all i) we know that $S_\tau - S_\sigma = \sum_{i=1}^K X_i 1_{\sigma < i \leq \tau}$. So $E(S_\tau) - E(S_\sigma) = E(S_\tau - S_\sigma) = \mu \sum_{i=1}^K E(1_{\sigma < i \leq \tau})$ as X_i is i.i.d. and independent of $1_{\sigma < i \leq \tau}$.

Applying Definition 7.29 (Notes 5/2), for a martingale we need this expression to be zero which is true iff $\mu = 0$, as $\sum_{i=1}^K E(1_{\sigma < i \leq \tau}) \geq 0$ with the equality being strict if $\sigma < \tau$.

b) Which kind of martingale property hold if $\mu \neq 0$?

Arguing with the same equation as before:

- If $\mu < 0$, then we have a submartingale as $E(S_\tau) - E(S_\sigma) \leq 0$.
- If $\mu < 0$, then we have a supermartingale as $E(S_\tau) - E(S_\sigma) \geq 0$.

Easy 29 (Test Feb-05), *7.37 (Notes Feb-01)* ✓

PROBLEM: Let $S_n = X_1 + X_2 + \cdots + X_n$ where (X_i) are independent identically distributed (i.i.d.) and integrable random variables with $E(X_i) = 0$. Let (H_k) be a predictable (with respect to the history of (S_n)) sequence of integrable random variables. Show that

$$V_n := V_0 + \sum_{k=1}^{n} H_k(S_k - S_{k-1})$$

is a martingale.

By Theorem 7.33 (notes 5/2), V_n being a martingale is equivalent to $E\left(V_n|\mathcal{F}_{n-1}\right) = V_{n-1}$, where $\mathcal{F}_n$ is the σ-field representing the history of (S_n).

By definition of S_n, $S_k - S_{k-1} = X_k$.

$$\begin{aligned}
E\left(V_n|\mathcal{F}_{n-1}\right) &= E\left(V_0 + \sum_{k=1}^{n} H_k(S_k - S_{k-1})|\mathcal{F}_{n-1}\right) \\
&= E\left(V_0 + \sum_{k=1}^{n-1} H_k X_k|\mathcal{F}_{n-1}\right) + E\left(H_n X_n|\mathcal{F}_{n-1}\right) \\
&= E\left(V_{n-1}|\mathcal{F}_{n-1}\right) + E\left(H_n X_n|\mathcal{F}_{n-1}\right) \\
&= V_{n-1} + H_n E\left(X_n\right) = V_{n-1}.
\end{aligned}$$

Using that V_{n-1} is $\mathcal{F}_{n-1}$-measurable, that H_n is predictable with respect to the history of (S_n) and that X_n is independent of $\mathcal{F}_{n-1}$ (as the X_i are i.i.d.). So we can use the Law of redundant conditioning.

Easy 30 (Test Feb-05), *8.4 (Notes Feb-01)* ✓

PROBLEM: Let $(W_t)_{t\geq 0}$ be a Wiener process. Show that $X_t := -W_t$, $t \geq 0$, is a Wiener process, too.

——

Show that $X_t := -W_t$, t ≥ 0 is a Wiener process.

Proof: We just need to verify the axioms stated in 8.1

(1) , $X_0 = -W_0 = 0$

(2) The increment $X_t - X_s = -W_t + W_s = -(W_t - W_s)$

Since we know, for s<t , $W_t - W_s \sim N(0, t-s)$ and is mutually independent for non-overlapping intervals, $-(Wt - Ws) \sim N(0, t-s)$ and mutually independent for non-overlapping intervals too.

(3) Continuity of all paths (P-a.s) $\forall \omega \in \Omega$

This follows that, if W_t is continuous and g(x)=-x is a continuous function then $X_t := g(W_t) = W_t$ is continuous by composition law. So we have done.

Easy 31 (Test Feb-05), *see Thm 8.18 (Notes Feb-01)*

PROBLEM: Let $(W_t)_{t\geq 0}$ be a Wiener process. Show that $(W_t)_{t\geq 0}$ and $X_t := W_t^2 - t$ are martingales w.r.t the history of $(W_t)_{t\geq 0}$.

——

First part: Show that $(W_t)_{t\geq 0}$ is a martingale w.r.t. its history,
i.e. $E(W_t|\mathcal{F}_s) = W_s$ $\forall s < t$.

Take s, t arbitrary such that $0 \geq s < t$.

$E(W_t|\mathcal{F}_s) = E((W_t - W_s) + W_s|\mathcal{F}_s)$ by simple albegra

$= E(W_t - W_s|\mathcal{F}_s) + E(W_s|\mathcal{F}_s)$ by linearity of E

$= E(W_t - W_s|\mathcal{F}_s) + W_s$ because W_s is $\mathcal{F}_s$-measurable

$= E(W_t - W_s + W_s$ since increments are independent of $\mathcal{F}_s$

$= 0 + W_s$ since increments have expectation zero.

Since s, t were chosen arbitrarily, we have therefore $E(W_t|\mathcal{F}_s) = W_s\ \forall s < t$. QED

Second part: Show that $X_t := W_t^2 - t$ is a martingale w.r.t. the history of $(W_t)_{t\geq 0}$ i.e. $E(W_t^2 - t|\mathcal{F}_s) = W_s^2 - s\ \forall s < t$.

First note that by simple algebra

$(W_t - W_s)^2 = W_t^2 - 2W_tW_s + W_s^2 = W_t^2 - 2W_tW_s + W_s^2 + W_s^2 - W_s^2 =$

$= (W_t^2 - W_s^2) - 2W_s(W_t - W_s)$.

By linearity of E,

$E((W_t - W_s)^2|\mathcal{F}_s) = E(W_t^2|\mathcal{F}_s) - E(W_s^2|\mathcal{F}_s) - 2E(W_s(W_t - W_s)|\mathcal{F}_s)$

So we have

$E(W_t^2|\mathcal{F}_s) - E(W_s^2|\mathcal{F}_s) = E((W_t - W_s)^2|\mathcal{F}_s) + 2E(W_s(W_t - W_s)|\mathcal{F}_s)$

Due to redundant conditioning

$2E(W_s(W_t - W_s)|\mathcal{F}_s) = 2W_sE(W_t - W_s|\mathcal{F}_s) = 2W_s0 = 0$

since increments are independent of $\mathcal{F}_s$.

From the independence of the increments of $\mathcal{F}_s$, we further get that the square of the increments must be independent of $\mathcal{F}_s$ as well, and thus we can write $E((W_t - W_s)^2|\mathcal{F}_s) = E((W_t - W_s)^2)$.

Using $Var(X) = E(X^2) - (E(X))^2$ we can write
$E((W_t - W_s)^2) = Var(W_t - W_s) + (E(W_t - W_s))^2 = (t - s) + 0$
by the properties of the Wiener Process.

Putting all this together, we get
$E(W_t^2|\mathcal{F}_s) - E(W_s^2|\mathcal{F}_s) = (t - s)$ or $E(W_t^2|\mathcal{F}_s) - W_s^2 = (t - s)$
since W_s^2 is $\mathcal{F}_s$-measurable because $(W_t)_{t\geq 0}$ is adapted to its history. By linearity of E and since t is deterministic, we get $E(W_t^2 - t|\mathcal{F}_s) = W_s^2 - s$ QED

Easy 32 (Test Feb-05), *see Thm 8.18 (Notes Feb-01)* ✓

> PROBLEM: Let $(W_t)_{t\geq 0}$ be a Wiener process. Show that $X_t := \exp(aW_t - a^2t/2)$ is a martingale w.r.t the history of $(W_t)_{t\geq 0}$.

——

Show that $X_t := e^{aW_t - a^2t/2}$ is a martingale w.r.t. the history of $(W_t)_{t\geq 0}$
i.e. $E(e^{aW_t - a^2t/2}|\mathcal{F}_s) = e^{aW_s - a^2s/2}$ $\forall s < t$.

By simple algebraic transformations we get
$E(e^{aW_t}|\mathcal{F}_s) = E(e^{a(W_t - W_s) + aW_s}|\mathcal{F}_s) = E(e^{a(W_t - W_s)}e^{aW_s}|\mathcal{F}_s)$.
Since W_s is $\mathcal{F}_s$-measurable, redundant conditioning leads to
$E(e^{a(W_t - W_s)}e^{aW_s}|\mathcal{F}_s) = e^{aW_s}E(e^{a(W_t - W_s)}|\mathcal{F}_s)$

Knowing that increments of the Wiener Process are independent of the past, and that for a random variable $X \sim N(0, \sigma^2)$ we have $E(e^{\lambda X}) = e^{\lambda^2\sigma^2/2}$ we obtain
$e^{aW_s}E(e^{a(W_t - W_s)}|\mathcal{F}_s) = e^{aW_s}E(e^{a(W_t - W_s)}) = e^{aW_s}e^{a^2(t-s)/2}$

Using the above results and applying simple algebraic transformations and the linearity of E, we get

$E(e^{aW_t}|\mathcal{F}_s) = e^{aW_s}e^{a^2t/2}e^{-a^2s/2}$

$\Rightarrow e^{-a^2t/2}E(e^{aW_t}|\mathcal{F}_s) = e^{aW_s-a^2s/2}$

$\Rightarrow E(e^{aW_t-a^2t/2}|\mathcal{F}_s) = e^{aW_s-a^2s/2}$ $\forall s,t$ such that $s<t$. QED

Easy 33 (Test Feb-05), *8.30 (Notes Feb-01)* ✓

Problem: Let $(W_t)_{t\geq 0}$ be a Wiener process and let $\tau_{a,b}$ be the first passage time of the boundary $f(t) = a + bt$, $a > 0$. Take for granted that

$$E(e^{-\lambda\tau_{a,b}}1_{(\tau_{a,b}<\infty)}) = e^{-a(b+\sqrt{b^2+2\lambda})},\ \lambda \geq 0$$

(a) Find $P(\tau_{a,b} < \infty)$.

(b) Find $E(\tau_{a,b})$.

——

We have $\mathrm{E}(e^{-\lambda\tau_{a,b}}.1_{(\tau_{a,b}<\infty)}) = e^{-a(b+\sqrt{b^2+2\lambda})}$, $\lambda \geq 0$

(a) Find $\mathrm{P}(\tau_{a,b} < \infty)$

Solution: We can set $\lambda = -\theta b + \frac{\theta^2}{2} = 0$ (we are free to choose θ)

Then we have $\mathrm{E}(1_(\tau_{a,b} < \infty)) = e^{-a(b+|b|)}$

Plus $E(1_{(\tau_{a,b}<\infty)}) = P(\tau_{a,b} < \infty)$

Hence $P(\tau_{a,b} < \infty) = e^{-a(b+|b|)}$

$(i), b < 0, P(\tau_{a,b} < \infty) = e^{-a(b-b)} = 1$

$(ii), b = 0, P(\tau_{a,b} < \infty) = e^{-a.0} = 1$

$(iii) b > 0, P(\tau_{a,b} < \infty) = e^{-2ab} < 1$ $(\forall a, b > 0)$

(b) Find $E(\tau_{a,b})$

Solution: (i) if $b > 0, P(\tau_{a,b} < \infty) = e^{-2ab} < 1 \Rightarrow \tau_{a,b} = \infty$ with some strictly positive probability $\Rightarrow E(\tau_{a,b} = \infty)$

(ii) Let $\tau_{a,b} < \infty$, firstly, we have $1_{(\tau_{a,b}<\infty)}(\omega) = 1\ \forall\omega \in \Omega$
Hence $\mathrm{E}(e^{-\lambda\tau_{a,b}}.1) = e^{-a(b+\sqrt{b^2+\lambda})}$

We differentiate both sides wrt λ (assume we can introduce E(.) and differentiation)

$\mathrm{E}(\tau_{a,b}e^{-\lambda\tau_{a,b}}) = e^{-a(b+\sqrt{b^2+\lambda})}.(\frac{a}{\sqrt{b^2+2\lambda}})$

Then set λ=0 , $\mathrm{E}(\tau_{a,b}) = e^{-a(b+|b|)}.\frac{a}{|b|} = \begin{cases} \infty, b = 0 \\ \frac{a}{|b|}, b < 0 \end{cases}$

Easy 34 (Test Feb-05), *8.35 (Notes Feb-01)* ✓

PROBLEM: Find the distribution of $\max_{s\leq t} W_s$. (Apply the formula for the distribution function of $\tau_{a,b}$.)

——

Prof. Strasser: *You did not use the hint, but it is OK*

Find the distribution of $\max_{s\leq t} W_s$. (Apply the formula for the distribution function of $\tau_{a,b}$.)

Proof: Use the reflection principle (although not subject of the exam). If $x = 0$,

$$P[\max_{s\leq t} W_s \geq y, W_t \leq y] = P(W_t \geq y)$$

$$P(\max_{s\le t} W_s \ge y) = P(\max_s W_s \ge y, W_t \le y) + P(\max_s W_s \ge y, W_t > y)$$
$$= P(\max_s W_s \ge y, W_t \le y) + P(W_t > y)$$
$$= 2P(W_t \ge y)$$

Then the distribution of the max is

$$P(\max_{s\le t} W_s \le y) = 1 - 2P(W_t \ge y) = 1 - 2(1 - \phi(\frac{y}{\sqrt{t}}))$$

Easy 35 (Test Feb-05), *8.36b (Notes Feb-01)* ✓

PROBLEM: Let $(W_t)_{t\ge 0}$ be a Wiener process, $c, d > 0$ and define

$$\sigma_{c,d} = \inf\{t : W_t \notin (-c, d)\}$$

Show that $P(\sigma_{c,d} < \infty) = 1$.

——

$c, d > 0$. Define
$\delta_{c,d}$:=inf$\{f : W_t \notin (-c, d)\}$

Show that P$(\delta_{c,d} < \infty) = 1$

Proof: $\delta_{c,d}$:=inf $\{t : W_t \ge d, or W_t \le -c\}$
=inf $\{t : W_t \ge d\} \bigcup inf\{t : -W_t \ge c\}$
$= \delta_{d,0} \bigcup \tau_{c,0}$

Where $\tau_{a,b} := inf\{t : W_t \ge a + bt\}$
Set b=0 , we get horizontal boundaries
Hence $\{\delta_{c,d} < \infty\} = \{\tau_{d,0} < \infty\} \bigcup \{\tau_{c,0} < \infty\}$

$\Rightarrow \{\tau_{d,0} < \infty\} \subseteq \{\delta_{c,d} < \infty\}$

$\Rightarrow P(\tau_{d,0} < \infty) \leq P(\delta_{c,d} < \infty)$

Since from #33 we have showed that $P(\tau_{c,d} < \infty) = 1$ if b=0

we end up with $P(\delta_{c,d} < \infty) \geq P(\tau_{d,0} < \infty) = 1$

However $P(\delta_{c,d} < \infty) \leq 1$

in the end $P(\delta_{c,d} < \infty) = 1$

2 Intermediate Questions

Intermediate 1 (Test Feb-05), *1.9 (Notes Feb-01)* ✓

PROBLEM: Let $\Omega = (-\infty, \infty]$ and let $\mathcal{R}$ be the system of subsets arising as unions of finitely many intervals of the form $(a, b]$ where $-\infty \leq a < b \leq \infty$ (left-open and right-closed intervals). Show that each element $B \in \mathcal{R}$ can be written as a union of disjoint intervals

$$B = \bigcup_{i=1}^{n} (a_i, b_i] \tag{4}$$

where $-\infty \leq a_1 < b_1 \leq a_2 < b_2 \leq a_3 < \ldots < b_{n-1} \leq a_n < b_n \leq \infty$.

Prof Strasser: *OK, induction arguments can be omitted*

a $\mathcal{H} := \left\{ H_i = \bigcup_{j=1}^{n} I_{ij}, \quad \text{where } I_{is} \cap I_{it} = \phi \ \forall s \neq t \right\} \subseteq \mathcal{R}$ where I_{ij} is defined as interval of type $(a, b]$ with $-\infty \leq a \leq b \leq \infty$ (so I_{ij} can be equal to the empty set)

Now we want to show that $\mathcal{H}$ is closed under intersections: $\bigcap_{i=1}^{m} H_i = \bigcap_{i=1}^{m} \bigcup_{j=1}^{n} I_{ij} \in \mathcal{H}$.

proof: Consider an arbitrary $w \in \bigcap_{i=1}^{m} \bigcup_{j=1}^{n} I_{ij} \in \mathcal{H}$. That such a w exists (if the intersection is empty, then we are done as the empty set is also an interval) means that $\forall i\, \exists j_i$ s.t. $w \in I_{ij_i} \quad \Leftrightarrow \quad \exists j_1, \ldots, j_m$ s.t. $w \in \bigcap_{i=1}^{m} I_{ij_i} \quad \Leftrightarrow \quad w \in \bigcup_{j_1,\ldots,j_m} \bigcap_{i=1}^{m} I_{ij_i}$.

So we have that $\bigcap_{i=1}^{m} H_i = \bigcap_{i=1}^{m} \bigcup_{j=1}^{n} I_{ij} = \bigcup_{j_1,\ldots,j_m} \bigcap_{i=1}^{m} I_{ij_i}$.

- It's obvious that $\bigcap_{i=1}^{m} H_i$ is a union of intervals of the form $(a_i, b_i]$ or the empty set. This is because $\bigcap_{i=1}^{m} I_{ij_i}$ is the intersection of m intervals and the intersection of intervals has to be an interval (or the empty set) again (trivial for $m = 2$ and then extended by induction).

- These intervals are pairwise disjoint, i.e. $(j_1, \ldots, j_m) \neq (k_1, \ldots, k_m) \Rightarrow \bigcap_{i=1}^m I_{ij_i} \cap \bigcap_{i=1}^m I_{ik_i} = \phi$.

  Just let $j_1 \neq k_1$. $\bigcap_{i=1}^m I_{ij_i} \subseteq I_{1j_1}$ and $\bigcap_{i=1}^m I_{ik_i} \subseteq I_{1j_1}$. By assumption we have $I_{is} \cap I_{it} = \phi \ \forall s \neq t$ and hence $I_{1j_1} \cap I_{1k_1} = \phi$.

b Now we want to show that every finite union of intervals can be written as

$$\bigcup_{i=1}^n I_i = I_1 \cup (I_2 \backslash I_1) \cup (I_3 \backslash (I_1 \cup I_2)) \cup \ldots \cup (I_n \backslash (I_1 \cup I_2 \cup \ldots \cup I_{n-1}))$$

where the k^{th} set is $(I_k \backslash \bigcup_{j=1}^{k-1} I_j) \in \mathcal{H}$ as $\mathcal{H}$ is closed under intersection and $(I_k \backslash \bigcup_{j=1}^{k-1} I_j) = (I_k \cap (\bigcup_{j=1}^{k-1} I_j)^c) = (I_k \cap (\bigcap_{j=1}^{k-1} I_j^c))$ is a finite intersection of intervals.

proof by induction:

n=2: $\bigcup_{i=1}^n I_i = I_1 \cup I_2$ and $I_1 \cup (I_2 \backslash I_1) = I_1 \cup (I_2 \cup I_1^c) = (I_1 \cup I_2) \cap \Omega = I_1 \cup I_2$ (the second equality follows from the distributive law).

Now assume it holds for $n = k$ which means $\bigcup_{i=1}^k I_i = I_1 \cup \ldots \cup (I_k \backslash \bigcup_{j=1}^{k-1} I_j)$. When $n = k+1$, we have

$$\begin{aligned}
& I_1 \cup (I_2 \backslash I_1) \cup (I_3 \backslash (I_1 \cup I_2)) \cup \ldots \cup (I_n \backslash (I_1 \cup I_2 \cup \ldots \cup I_{n-1})) \\
= \ & I_1 \cup \ldots \cup \left(I_k \backslash \bigcup_{j=1}^{k-1} I_j \right) \cup \left(I_{k+1} \backslash \bigcup_{j=1}^{k} I_j \right) \\
= \ & \bigcup_{j=1}^{k} I_j \cup \left(I_{k+1} \cap (\bigcup_{j=1}^{k} I_j)^c \right) \\
= \ & \bigcup_{j=1}^{k} I_j \cup I_{k+1} \cap \left(\bigcup_{j=1}^{k} I_j \cup (\bigcup_{j=1}^{k} I_j)^c \right) \\
= \ & \bigcup_{j=1}^{k+1} I_j \cap \Omega = \bigcup_{j=1}^{k+1} I_j
\end{aligned}$$

where we apply the distributive law again.

The only thing left to show is that $(I_m \backslash \bigcup_{j=1}^{m-1} I_j) \cap (I_l \backslash \bigcup_{j=1}^{l-1} I_j) = \phi \quad \forall m \neq l$ (pairwise disjointness).

Let $m < l$. Then

$$\left(I_m \backslash \bigcup_{j=1}^{m-1} I_j\right) \cap \left(I_l \backslash \bigcup_{j=1}^{l-1} I_j\right) = \left(I_m \backslash \bigcup_{j=1}^{m-1} I_j\right) \cap \left(I_l \backslash ((\bigcup_{j=1}^{m-1} I_j) \cap I_m \cap \ldots \cap I_{l-1})\right)$$

$$= \left(I_m \cap \bigcap_{j=1}^{m-1} I_j^c\right) \cap \left(I_l \cap (\bigcup_{j=1}^{m-1} I_j^c) \cap I_m^c \cap \ldots \cap I_{l-1}^c\right) = \phi$$

as I_m is intersected with I_m^c.

Intermediate 2 (Test Feb-05), *1.17 (Notes Feb-01)* ✓

PROBLEM: Show that every finite field is generated by a partition.

——

let $\mathcal{R}$ be a finite field on Ω; so $\mathcal{R} = \{\phi, \Omega, R_1, \ldots, R_n\}$; for any $x \in \Omega$ define $A_x := \bigcap\{A \in \mathcal{R} : x \in A\}$

further, $A \in \mathcal{R}$ is an "atom" if $A \neq \phi$ and $\phi \neq B \subseteq A, B \in \mathcal{R} \Rightarrow B = A$.

to show $\forall x \in \Omega$, A_x is the unique atom containing x

- A_x is an atom: Suppose not, then $\exists B \in \mathcal{R}$ s.t. $x \in B$, $B \subseteq A_x$, $B \neq A$ and B an atom; by def. of A_x, $A_x \subseteq B$; that's a contradiction $\Rightarrow B = A_x \Rightarrow A_x$ is an atom

- A_x is unique atom containing x: Suppose not, then $\exists C \in \mathcal{R}$ s.t. $x \in C$ and C an atom; again by def. of A_x, $A_x \subseteq C$, but since C is an atom, $A_x = C \Rightarrow A_x$ is the unique atom containing x!

to show $\mathcal{C} = \{A_x : x \in \Omega\}$ is a partition of Ω and $\mathcal{C}$ generates $\mathcal{R}$!

- $\forall x \in \Omega$: $A_x \neq \phi$; obvious as $x \in A_x$ per definition

- Since the number of sets in $\mathcal{R}$ is finite, and A_x are intersections of these finite sets, the number of A_x must also be finite: $\mathcal{C} = \{A_{x_1}, \ldots, A_{x_m}\}$
- $\forall x \in \Omega$ we have $A_x \in \mathcal{R}$; obvious as A_x is a finite intersection of sets of $\mathcal{R}$ (by definition of a field)
- $\forall A_x$ we have $A_x \subseteq \Omega$: by def. of A_x and fact that $\Omega \in \mathcal{R}$ we have that $\Omega = \bigcup_{x\in\Omega}\{x\} \subseteq \bigcup_{x\in\Omega}\{A_x\} \subseteq \Omega \Rightarrow \bigcup_{x\in\Omega}\{A_x\} = \Omega$
- for any $A_{x_i} \neq A_{x_j}$ we have $(A_{x_i} \cap A_{x_j}) = \phi$: Suppose that $(A_{x_i} \cap A_{x_j}) \neq \phi$. Then $\exists z \in \Omega$ s.t. $z \in (A_{x_i} \cap A_{x_j})$. So it must hold that for $A_z = \bigcap\{A \in \mathcal{R} : z \in A\}$, $A_z \subseteq A_{x_i}$ and $A_z \subseteq A_{x_j}$. Since A_z, A_{x_j} and A_{x_i} are atoms, this implies $\phi \neq A_z = A_{x_j}$ and $\phi \neq A_z = A_{x_j}$. From there we can deduce that $A_{x_i} = A_{x_j}$ which is a contradiction.

so $\mathcal{C} = \{A_x : x \in \Omega\}$ is a partition of Ω containing m elements as we have $A_x \neq \phi \, \forall x \in \Omega$, $\quad \bigcup_{x\in\Omega}\{A_x\} = \Omega \quad$ and $\quad (A_{x_i} \cap A_{x_j}) = \phi \, \forall A_{x_i} \neq A_{x_j}$.

For any $R_i \in \mathcal{R}$ we can write: $R_i = \bigcup_{i\in\alpha_i} A_{x_i}$ where $\alpha_i : \Omega \to (1, \ldots, m)$ and $\alpha_i(x) = \{j : x \in A_{x_j}\}$

- any element of $\mathcal{R}$ can be written as the finite union of elements of $\mathcal{C}$
- $\mathcal{C} = \{A_x : x \in \Omega\}$ generates the finite field $\mathcal{R}$ ($\mathcal{C} = \{A_{x_1}, \ldots, A_{x_m}\}$

Intermediate 3 (Test Feb-05), *1.21 (Notes Feb-01)* ✓

Problem: (a) A field $\mathcal{F}$ is a σ-field iff

$$(F_i)_{i\in\mathbb{N}} \subseteq \mathcal{F} \;\Rightarrow\; \bigcap_{i\in\mathbb{N}} A_i \in \mathcal{F}$$

(b) A field $\mathcal{F}$ is a σ-field iff the union of every increasing (decreasing) sequence of sets in $\mathcal{F}$ is in $\mathcal{F}$, too.

(c) A field $\mathcal{F}$ is a σ-field iff the union of every pairwise disjoint sequence of sets in $\mathcal{F}$ is in $\mathcal{F}$, too.

(a) Suppose $\mathcal{F}$ is a σ-field. Then all $A_i^C \in \mathcal{F}$, $\bigcup_{i\in\mathbf{N}} A_i^C \in \mathcal{F}$ and by de Morgan's laws

$$\bigcap_{i\in\mathbf{N}} A_i = \left(\bigcup_{i\in\mathbf{N}} A_i^C\right)^C \in \mathcal{F}.$$

Now suppose $(A_i)_{i\in\mathbf{N}} \subseteq \mathcal{F} \Rightarrow \bigcap_{i\in\mathbf{N}} A_i \in \mathcal{F}$. Now take any $(A_i)_{i\in\mathbf{N}} \subseteq \mathcal{F}$. By nearly the same arguments (the complements are in $\mathcal{F}$, because $\mathcal{F}$ is is field)

$$\bigcup_{i\in\mathbf{N}} A_i = \left(\bigcap_{i\in\mathbf{N}} A_i^C\right)^C \in \mathcal{F}.$$

(b) Suppose $\mathcal{F}$ is a σ-field. Then for $(A_i)\uparrow$, $(A_i)_{i\in\mathbf{N}}$ trivially $\bigcup_{i\in\mathbf{N}} A_i \in \mathcal{F}$ holds.

Now suppose that the union of every increasing sequence of sets in $\mathcal{F}$ is in $\mathcal{F}$ too. Suppose we have $(A_i)_{i\in\mathbf{N}} \subseteq \mathcal{F}$. Then $B_i := \bigcup_{n\leq i} A_n$ is an increasing sequence, but then

$$\bigcup_{i\in\mathbf{N}} A_i = \bigcup_{i\in\mathbf{N}} B_i \in \mathcal{F}.$$

(c) Suppose $\mathcal{F}$ is a σ-field. Then for (A_i) p.w.-disjoint trivially $\bigcup_{i\in\mathbf{N}} A_i \in \mathcal{F}$ holds.

Now suppose, that the union of every p.w.-disjoint sequence of sets in $\mathcal{F}$ is in $\mathcal{F}$ too. Let $(A_i)_{i\in\mathbf{N}} \subseteq \mathcal{F}$. Define $B_i := A_i \setminus \bigcup_{n<i} A_n \in \mathcal{F}$ (because $\mathcal{F}$ is a field). But the B_i are p.w.-disjoint. Hence

$$\bigcup_{i\in\mathbf{N}} A_i = \bigcup_{i\in\mathbf{N}} B_i \in \mathcal{F}.$$

Intermediate 4 (Test Feb-05), *2.7 (Notes Feb-01)* ✓

PROBLEM: Let $f : (\Omega, \mathcal{A}) \to (Y, \mathcal{B})$ be $(\mathcal{A}, \mathcal{B})$-measurable, and let $g : (Y, \mathcal{B}) \to (Z, \mathcal{C})$ be $(\mathcal{B}, \mathcal{C})$-measurable. Then $g \circ f$ is $(\mathcal{A}, \mathcal{C})$-measurable.

——

we need to show that $(g \circ f)^{-1}(C) \in \mathcal{A} \quad \forall C \in \mathcal{C}$.
Since f is $(\mathcal{A}, \mathcal{B})$-measurable, g is $(\mathcal{B}, \mathcal{C})$-measurable, we have
$f^{-1}(B) \in \mathcal{A}\ \forall B \in \mathcal{B}$ and $g^{-1}(C) \in \mathcal{B}\ \forall C \in \mathcal{C}$

So

$$\begin{aligned}(g \circ f)^{-1}(C) &= f^{-1}(g^{-1}(C)) \qquad \forall C \in \mathcal{C} \\ &= f^{-1}(B) \in \mathcal{A} \qquad \forall B := g^{-1}(C) \in \mathcal{B}\end{aligned}$$

Intermediate 5 (Test Feb-05), *2.11 (Notes Feb-01)* ✓

PROBLEM: Show that a function $f : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable iff $(f \leq \alpha) \in \mathcal{F}$ for every $\alpha \in \mathbb{R}$.

——

$(f \leq \alpha) = f^{-1}[(-\infty, \alpha]]$

proof of $\Rightarrow$ clear by definition of measurability, since $(-\infty, \alpha] \in \mathcal{B}(\mathbf{R})\ \forall \alpha \in \mathbf{R}$

proof of $\Leftarrow$ $(f \leq \alpha) \in \mathcal{F}\ \forall \alpha \in \mathbf{R}$, i.e. $f^{-1}[(-\infty, \alpha]] \in \mathcal{F}\ \forall \alpha \in \mathbf{R}$
Now take $a, b \in \mathbf{R}$ with $a < b$:

$$f^{-1}[(-\infty, a]] \in \mathcal{F}, \qquad f^{-1}[(-\infty, b]] \in \mathcal{F}$$

since $\mathcal{F}$ is a field we have:

$$\left[f^{-1}[(-\infty,a]]\right]^c \in \mathcal{F} \quad \text{and} \quad \left[f^{-1}[(-\infty,a]]\right]^c \cap \left(f^{-1}[(-\infty,b]]\right) \in \mathcal{F}$$

$$\begin{aligned}
&\Rightarrow \; f^{-1}[(-\infty,a]^c] \cap \left(f^{-1}[(-\infty,b]]\right) \in \mathcal{F} \\
&\Rightarrow \; f^{-1}\left[(-\infty,a]^c \cap (-\infty,b]\right] \in \mathcal{F} \\
&\Rightarrow \; f^{-1}[(a,b]] \in \mathcal{F} \qquad \forall a,b \in \mathcal{R};\; a<b
\end{aligned}$$

$\mathcal{R} = \{$finite unions of intervals $(a,b]$, $-\infty \le a < b \le \infty\}$

for any $R \in \mathcal{R}:\quad R = \bigcup_{i=1}^n (a_i,b_i]$

since $\mathcal{F}$ is a field:

$$\bigcup_{i=1}^n f^{-1}\left[(a_i,b_i]\right] \in \mathcal{F} \quad \Rightarrow \quad f^{-1}\left[\bigcup_{i=1}^n [(a_i,b_i]]\right],$$

i.e. $f^{-1}(R) \in \mathcal{F} \quad \forall R \in \mathcal{R}$

$\mathcal{R}$ generates the σ-Borel-function $\mathbf{R}$, i.e. $\sigma(\mathcal{R}) = \mathcal{B}(\mathbf{R})$

$\Rightarrow$ therefore, by Theorem 2.8 (as of 5/2), f is $\mathcal{F}$-measurable!

Intermediate 6 (Test Feb-05), *3.14 (Notes Feb-01)* ✓

PROBLEM: Let $f \in \mathcal{L}(\mathcal{F})^+$. Prove Markoff's inequality

$$\mu(f>a) \le \frac{1}{a}\int f\,d\mu, \quad a>0.$$

——

Prof. Strasser: *OK, but the contrary of $>$ is $\le$!*

Let $f \in (\mathcal{F})^+$. Prove Markoff's inequality

$\mu\,(f>a) \le \frac{1}{a}\int f d\mu, a>0$

Proof:

$f \geq a1_{\{f>a\}}$, because $1\left\{_{f>a}\right\} = \left[\begin{array}{l} 1, \text{ when } f > a \\ 0, \text{ when } f < a \end{array}\right.$

$\Rightarrow \int f d\mu \geq \int a1\left\{_{f>a}\right\} d\mu = \mu\left(f > a\right)$

$\Updownarrow$

$\frac{1}{a} \int f d\mu \geq \mu\left(f > a\right)$

Intermediate 7 (Test Feb-05), *3.15 (Notes Feb-01)* ✓

PROBLEM: Let $f \in \mathcal{L}(\mathcal{F})^+$. Show that $\int f\, d\mu = 0$ implies $\mu(f \neq 0) = 0$.

——

f $\in L(F)^{-1}$, Show $\int f d\mu = 0 \Rightarrow \mu(f \neq 0) = 0$

Let $\int f d\mu = 0$. From Markov's inequality and since f $\in L(F)^t$ then

$$0 \leq \mu(f > \frac{1}{n}) \leq \frac{1}{\frac{1}{n}} \int f d\mu$$

(call it inequality 1)

Now consider the sets $A_n = (f > \frac{1}{n})$. We have

$A_1 \subset A_2 \subset A_3 \subset ...and A_n \uparrow A_0 = (f > 0)$ as $n \to \infty$

Hence , we have $lim_n A_n = \bigcup_{n \in N} A_n = A_0$

Then by σ-additivity

$\mu(\bigcup_n A_n) = \mu(A_0)$

Now as n$\to \infty$ inequality 1 becomes

$0 \leq \mu(f > 0) \leq 0$

The second inequality follows since $\int f d\mu$=0.

Hence $\mu(f > 0) = \mu(f \neq 0) = 0 \Rightarrow f = 0(\mu - a.e)$

Intermediate 8 (Test Feb-05), *3.26b (Notes Feb-01)* ✓

PROBLEM: Let f be an integrable function. Then

$$f = 0\,\mu\text{-a.e.} \;\Leftrightarrow\; \int_A f\,d\mu = 0 \text{ for all } A \in \mathcal{F}$$

Let $A \in \mathcal{F}$ be arbitrary, but fixed. First consider $f = \sum_{i=1}^n a_i 1_{F_i} \in \mathcal{S}^+$, where $a_i > 0$. By $f = 0$ μ-a.e., ie. $\mu(f > 0) = 0$, we know that $\mu(F_i) = 0$ and hence $\mu(F_i \cap A) = 0$ for all i. But then

$$\int_A f\,d\mu = \int 1_A f\,d\mu = 0\,.$$

Now let $f \in \mathcal{L}^+$. Take a sequence $f_n \uparrow f$ where $f_n \in \mathcal{S}^+$ for all n. But $f = 0$ μ-a.e., hence $f_n = 0$ μ-a.e.. Plugging into the definition of the integral and using the result for simple functions we get

$$0 = \lim_{n\to\infty} \int_A f_n\,d\mu = \int_A f\,d\mu\,.$$

In the last step we consider $f \in \mathcal{L}$. We know that $f^+ = 0$ μ-a.e. and $f^- = 0$ μ-a.e., hence

$$\int_A f\,d\mu = \int_A f^+\,d\mu - \int_A f^-\,d\mu = 0\,.$$

Now we prove the converse. Suppose $\mu(f \neq 0) > 0$. Then either $\mu(f > 0) > 0$ or $\mu(f < 0) > 0$ holds. We only consider the first case: We claim there exists an $\varepsilon > 0$ such that $\mu(f > \epsilon) > 0$. Suppose not, then for every $\varepsilon > 0$ we have $\mu(f > \epsilon) = 0$. But $\{f > \frac{1}{n}\} \downarrow \{f > 0\}$ hence $0 = \lim_{n\to\infty} \mu(f > \frac{1}{n}) = \mu(f > 0) \neq 0$.

Let $A := \{f > 0\}$. Then we get a contradiction:

$$\int_A f\,d\mu = \int_{f>\varepsilon} f\,d\mu + \int_{0<f<\varepsilon} f\,d\mu \geq \varepsilon\,\mu(\{f > \varepsilon\}) > 0$$

Intermediate 9 (Test Feb-05), *3.32 (Notes Feb-01)* ✓

> PROBLEM: Show that under the assumptions of the dominated convergence theorem we even have
>
> $$\lim_n \int |f_n - f| \, d\mu = 0$$

—

$f_n \to f$ μ-a.e., therefore $h_n := |f_n - f| \to 0$ μ-a.e.. By the construction of h_n we know

$$h_n = |f_n - f| \leq |f_n| + |f| \leq g + |f|$$

By the dominated convergence theorem for f_n we know that f is integrable, hence h_n is dominated by the integrable function $g + |f|$. Again applying the dominated convergence theorem, but now for h_n, we get that $\lim_n h_n \in \mathcal{L}^1(\mu)$ and together with exercise 3.22

$$\lim_n \int h_n \, d\mu = \lim_n \int |f_n - f| \, d\mu = 0 \, .$$

Intermediate 10 (Test Feb-05), *4.3a (Notes Feb-01)* ✗

> Prof. Strasser: *far too complicated. you don't need the CS inequality. simply use (f+g)^2\le 2(f^2+g^2). derive this from (f+g)^2\ge 0.*
>
> PROBLEM: Show that $\mathcal{L}^2$ is a vector space.

—

Show that $\mathcal{L}^2 = \mathcal{L}^2(\Omega, \mathcal{F}, \mu) = \{f \in \mathcal{F} : \int f^2 d\mu < \infty\}$ is a vector space.

We first introduce the Cauchy-Schwarz-Inequality that we will use later: We know that $\langle x, x\rangle \geq 0$ for any $x \in \mathcal{L}^2$. Take $x = u + \lambda$. In addition we know that $\langle \lambda u, v\rangle = \lambda * \langle u, v\rangle \;\; \forall u, v \in \mathcal{L}^2, \lambda \in \mathcal{F}$ then

$$\langle u + \lambda v, u + \lambda v\rangle \; = \langle u, u\rangle \, \lambda * \langle u, v\rangle \; + \lambda * \langle v, u\rangle \; + \lambda\lambda * \langle v, v\rangle \geq 0$$

For $v = 0$, the inequality is trivially fulfilled ; for $v \neq 0$ we can set $\lambda = -\frac{\langle v,u\rangle}{\langle v,v\rangle}$ which leads to

$$\langle u,u\rangle - \frac{\langle v,u\rangle}{\langle v,v\rangle}\langle u,v\rangle - \frac{\langle u,v\rangle}{\langle v,v\rangle} < \langle v,u\rangle + \frac{\langle v,u\rangle}{\langle v,v\rangle}\langle v,v\rangle \geq 0$$

multiplying by $\langle v,v\rangle\ > 0$ yields

$$\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle\langle v,u\rangle \geq 0$$

Using this result we get that

$$\langle f,g\rangle\langle g,f\rangle \leq \langle f,f\rangle\langle g,g\rangle$$

since both $\langle f,f\rangle$ and $\langle g,g\rangle$ are finite by assumption, also $\langle f,g\rangle\langle g,f\rangle = ,\langle f,g\rangle,^2$ and hence $\langle f,g\rangle$ has to be finite concluding that it is an element of $\mathcal{L}^2$.

Have $\langle\lambda f,\lambda f\rangle = \lambda\lambda * \langle f,f\rangle$. Since both, λ and $\langle f,f\rangle$ are finite, the right hand side is finite. Thus $\langle\lambda f,\lambda g\rangle < \infty$ and $\lambda f \in \mathcal{L}^2$.

Finally we show that $f+g \in \mathcal{L}^2$. This means we have to show that $\langle f+g,f+g\rangle < \infty$. Expanding the right hand side and using the Cauchy Schwarz inequality gives

$$\langle f+g,f+g\rangle = \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle$$

. Here, $\langle f,f\rangle$ and $\langle g,g\rangle$ are finite by assumption and from the part before we know that also $\langle f,g\rangle$ and $\langle g,f\rangle$ are finite. Thus we may conclude that $f+g \in \mathcal{L}^2$and $\mathcal{L}^2$ is indeed a vector space.

Intermediate 11 (Test Feb-05), *4.12 (Notes Feb-01)* ✗

Prof. Strasser: *OK, but the final implication you need the measure extension theorem*

Problem: Let $\alpha : \mathbb{R} \to \mathbb{R}$ be an increasing function which is supposed to be continuous on $\mathbb{R}$ and to not differentiable at at most finitely many points. Show that $\lambda_\alpha = \alpha'\lambda$.

——

Let λ_α be a right-continuous, increasing function defined on left-open, right-closed intervals such that

$$\lambda_\alpha\left((a,b]\right) = \alpha\left(b\right) - \alpha\left(a\right)$$

As α is increasing, continuous and differentiable (Except for finetely many points), α' is continuous almost everywhere.

$$\alpha\left(b\right) - \alpha\left(a\right) = \int_{(a,b]} \alpha' d\lambda = \int_a^b \alpha'(t)dt$$

where α' denotes the Radon-Nikodym-derivative (as in Definition 4.8 in the notes of 2006-02-05)

$$\alpha' = \frac{d\lambda_\alpha}{d\lambda}$$

So

$$\lambda_\alpha\left((a,b]\right) = \int_{(a,b]} \alpha' d\lambda = \left(\alpha'\lambda\right)\left((a,b]\right)$$
$$\Rightarrow \lambda_\alpha = \alpha'\lambda$$

Intermediate 13 (Test Feb-05), *4.13 (Notes Feb-01)* ✗

Prof. Strasser: *(a) you have to show that* $Q \ll P \equiv (P(C_i) = 0$ *implies* $Q(C_i) = 0$ *) (b) OK*

Problem: Let P and Q be probability measures of a finite field $\mathcal{F}$. (1) State $Q \ll P$ in terms of the generating partition of $\mathcal{F}$. (2) If $Q \ll P$ find dQ/dP.

——

Let P and Q be probability measures of a finite field $\mathcal{F}$

State $Q \ll P$ in terms of the generating partition of Ω

Let

$$\mathcal{C} = (C_1, C_2, ..., C_m)$$

be a finite partition of Ω

Then $\mathcal{F}$ is the σ -field generated by $\mathcal{C}$

$$\mathcal{F} = \sigma(C_1, C_2, ..., C_m)$$

Let $Q|\mathcal{F}$ and $P|\mathcal{F}$ be two probability measures defined on $(\Omega, \mathcal{F}, Q)$ and $(\Omega, \mathcal{F}, P)$

If $Q \ll P$:

then $P(A) = 0 \ \Rightarrow \ Q(A) = 0 \quad \forall A \in \mathcal{F}$

So:

$$P(A) = \sum_{C_i \subseteq A} P(C_i) = 0 \quad \text{by sigma-additivity}$$
$$\Rightarrow \quad P(C_i) = 0 \quad \forall\, C_i \subseteq A$$

Since $Q(A) = \sum_{C_i \subseteq A} P(C_i)$ and $Q(A) \geq 0 \quad \forall\, A \in \mathcal{F}$:

$$Q(C_i) = 0 \quad \forall\, C_i \subseteq A$$

and therefore

$$\sum_{C_i \subseteq A} Q(C_i) = Q(A) = 0$$

If $Q \ll P$, find dQ/dP

By the Radon-Nikodym-Theorem:

If $Q \ll P$, then $Q = fP$ for some $f \in \mathcal{L}^+(\mathcal{F})$

So the Radon-Nikodym-derivative of Q with respect to P is

$$f := \frac{dQ}{dP}$$

Moreover:

$$Q(A) = \int_A f\ dP = \int 1_A\ f\ dP$$

and:

$$Q(A) = \sum_{C_i \subseteq A} Q(C_i) = \sum_{C_i \subseteq A} \frac{Q(C_i)}{P(C_i)} P(C_i) = \sum_{i=1}^{m} 1_{C_i \cap A} \frac{Q(C_i)}{P(C_i)} P(C_i)$$

This implies:

$$\sum_{i=1}^{m} 1_{C_i \cap A} \frac{Q(C_i)}{P(C_i)} P(C_i) \; = \; \int 1_A \, f \, dP$$

then, as:

$$1_A f = \sum_{i=1}^{m} \frac{Q(C_i)}{P(C_i)} 1_{C_i \cap A}$$

and since $1_{C_i \cap A} = 1_A \, 1_C$:

$$\frac{dQ}{dP} =: f = \sum_{i=1}^{m} \frac{Q(C_i)}{P(C_i)} 1_{C_i}$$

Intermediate 14 (Test Feb-05), *4.15 (Notes Feb-01)* ✗

Prof. Strasser: *almost OK: you proved it for B in the field R. use the measure extension theorem to prove it for borel sets.*

Problem: Let $(\Omega, \mathcal{F}, P)$ be a measure space and X a random variable with differentiable distribution function F. Explain the formulas

$$P(X \in B) = \int_B F'(t)\, dt E(g \circ X) = \int g(t) F'(t)\, dt$$

$$P(X \in B) = \int_B F^{'}(t) dt :$$

From the fundamental theorem of calculus we know that $F(b) - F(a) = \int_a^b F^{'}(t)dt$. By definition of a distribution function we know that $P^X((a,b]) = F(b) - F(a) = \int_a^b F^{'}(t)dt$. X is a real-valued function $\Omega \to \mathbf{R}$, and the σ-field on $\mathbf{R}$ is the Borel-σ-field $\mathcal{B}(\mathbf{R})$ which is generated by the algebra $\mathcal{R}$. So by Problem 1.9 (Notes 5/2), B has to be the union of some pairwise disjoint intervals $B_i := (a_i, b_i]$.

So we have:

$$\begin{aligned} P(X \in B) &= \sum_i P(X \in B_i) = \sum_i [F(b_i) - F(a_i)] = \sum_i \int_{a_i}^{b_i} F^{'}(t)dt \\ &= \sum_i \int_{B_i} F^{'}(t)dt = \sum_i \int 1_{B_i} F^{'}(t)dt = \int \sum_i 1_{B_i} F^{'}(t)dt \\ &= \int 1_B F^{'}(t)dt = \int_B F^{'}(t)dt \end{aligned}$$

as $1_B = \sum_i 1_{B_i}$ (as the B_i are a generating partition).

$$\textit{and} \quad E(g \circ X) = \int g(t)F^{'}(t)dt :$$

By Problem 3.7 (Notes 5/2) we know that $E(g \circ X) = \int g \, d\lambda_F$. As F is differentiable, we know by Problem 4.12 (Notes 5/2) that $\lambda_F = F^{'}\lambda$. So $\int g \, d\lambda_F = \int g \, d(F^{'}\lambda) = \int g \, F^{'} d\lambda = \int_{-\infty}^{\infty} g(t)F^{'}(t)dt$.

Intermediate 15 (Test Feb-05), *6.8 (Notes Feb-01)* ✓

Problem: Prove the law of interated conditioning.

Correct proof by Prof. Strasser:

1. Assume that $\mathcal{A} \subseteq \mathcal{B}$. Then $E(X|\mathcal{A})$ is $\mathcal{B}$-measurable. Therefore we have

$$E(E(X|\mathcal{A})|\mathcal{B}) = E(X|\mathcal{A})$$

2. Assume conversely that $\mathcal{B} \subseteq \mathcal{A}$. In order to show that

$$E(E(X|\mathcal{A})|\mathcal{B}) = E(X|\mathcal{B})$$

we have to show that

$$\int_B E(X|\mathcal{B}) \, dP = \int_B E(X|\mathcal{A}) \, dP \text{ for all } B \in \mathcal{B}$$

But this is true since in view of $\mathcal{B} \subseteq \mathcal{A}$ both sides are equal to $\int_B X \, dP$.

Intermediate 16 (Test Feb-05), *6.10 (Notes Feb-01)* ✓

Problem: Prove the law of redundant conditioning.

——

Let X,Y be square integrable, X $\mathcal{A}$-measurable.

show that $E(xy|\mathcal{A})$=x $E(y|\mathcal{A})$

X,Y square-integrable $\Rightarrow$ X integrable, Y integrable $\Rightarrow$ XY integrable

1) Suppose that both X and Y are positive and let $A \in \mathcal{A}$

Suppose that X is a simple function $1_B = X$ for some $B \in \mathcal{A}$

$$\Rightarrow \int_A XE\left(Y|\mathcal{A}\right)dP = \int_{A\cap B} E\left(Y|\mathcal{A}\right) = \int_{A\cap B} YdP = \int_A XYdP$$

$\Rightarrow$ in this case we have

$$E\left(XY|\mathcal{A}\right) = xE\left(Y|\mathcal{A}\right)$$

Now we will use measure-theoretic induction:

X,Y are non-negative and X $\mathcal{A}$ measurable.

Let $\{X_n\}$ be an increasing sequence of simple $\mathcal{A}$-measurable functions with limit X.

Then X_n is simple.

$$\Rightarrow \int_A X_n E\left(Y|\mathcal{A}\right)dP = \int_A X_n YdP \quad (*)$$

Let $n \to \infty$, since $X \geq 0$, $E\left(x|\mathcal{A}\right) \geq 0$ $\left(x_n\right) \subseteq S(F)^+ X_n \uparrow$

we can apply the Beppo-Levi Theorem to (*) and obtain:

$$\int_A XE\left(y|\mathcal{A}\right)dP = \int_A XYdP \quad (**)$$

Now in the general case (when X, Y can also be negative), write X^+=max$\{X, 0\}$ and $X^- = X^+ - X = max\{-X, 0\}$

Then both X^+ and X^- are positive and both are $\mathcal{A}$-measurable. In a similar way we can define the positive and negative part of Y.

If x is square integrable $\Rightarrow X^+$, X^- are integrable. Moreover

$$XY = X^+X^+ + X^-Y^- - X^+Y^- - X^-Y^+$$

But (**) holds for all four products $\Rightarrow$ it holds for XY.

Intermediate 17 (Test Feb-05), *7.12 (Notes Feb-01)* ✓

PROBLEM: Let $(\mathcal{F}_k)$ be a filtration and let $\tau : \Omega \to \mathbb{N}_0 \cup \{\infty\}$ be a random variable. Show that the following assertions are equivalent:

(a) $(\tau = k) \in \mathcal{F}_k$ for every $k \in \mathbb{N}$

(b) $(\tau \leq k) \in \mathcal{F}_k$ for every $k \in \mathbb{N}$

(c) $(\tau < k) \in \mathcal{F}_{k-1}$ for every $k \in \mathbb{N}$

(d) $(\tau \geq k) \in \mathcal{F}_{k-1}$ for every $k \in \mathbb{N}$

(e) $(\tau > k) \in \mathcal{F}_k$ for every $k \in \mathbb{N}$

Let $(F_k)_{k\geq 0}$ be a filtration and $\tau : \Omega \to N_0 \bigcup \{\infty\}$ be a random variable.

(a) $\Rightarrow$ (b):

Since (τ=k) $\in F_k$ $\forall k \in$ N $\Rightarrow (\tau = k-1) \in F_{k-1}$.

Now we can write $(\tau \leq k) = \bigcup_{i=o}^{k} (\tau = i)$ where $(\tau = i) \in F_i$ $\forall i =$ 0,...,k .

Since (F_k) is a filtration, $F_{j-1} \subseteq F_j \forall j \in N_0) \Rightarrow$ $(\tau = i) \in F_i \subseteq F_k$ i=0,1,...,k. Since F is a field, it must contain the union of a finite number of its elements. Therefore, $(\tau \leq k) \in F_k$ $\forall$ $k \in N$.

(b) $\Rightarrow$(c):

Since $(\tau \leq k)$ $\in F_k \forall k \in$ $N \Rightarrow (\tau < k) = (\tau < k-1)$ $\in F_{k-1} \forall k \in N$.

(c) $\Rightarrow$(d):

F_{k-1} is a σ-field. Therefore we can conclude $(\tau < k) \in F_{k-1} \;\Rightarrow\; (\tau \geq k) = (\tau < k)^c \in F_{k-1}$.

(d) $\Rightarrow$(e):

Since $(\tau \geq k) \in F_{k-1} \forall k \in N \;\Rightarrow\; (\tau > k) = (\tau \geq k+1) \in F_k \forall k \in N$

(e) $\Rightarrow$(a):

We have $(\tau > k) \in F_k \forall k \in N$.

Now write $(\tau = k) = (\tau \leq k) \bigcap (\tau \geq k) = (\tau > k)^c \bigcap (\tau > k-1)$, where $(\tau > k)^c \;\in F_k$, and $(\tau > k-1) \in F_{k-1}$.

Since (F_k) is a filtration, $F_{k-1} \subseteq F_k$, and therefore $(\tau > k-1) \in F_k$.

Intermediate 18 (Test Feb-05), *7.36 (Notes Feb-01)* ✓

> Problem: Let $S_n = X_1 + X_2 + \cdots + X_n$ where (X_i) are independent identically distributed (i.i.d.) and square integrable random variables with $E(X_i) = 0$ and $V(X_i) = \sigma^2$. Show that $M_n := S_n^2 - \sigma^2 n$ is a martingale.

——

By Theorem 7.33 (Notes 5/2), M_n being a martingale is equivalent to $E\left(M_n | \mathcal{F}_{n-1}\right) = M_{n-1}$, where $\mathcal{F}_{n-1}$ is the σ-field representing the history of (S_n).

$$\begin{aligned}
E\left(M_n|\mathcal{F}_{n-1}\right) &= E\left(S_n^2 - \sigma^2 n|\mathcal{F}_{n-1}\right) \\
&= E\left(S_n^2|\mathcal{F}_{n-1}\right) - E\left(\sigma^2 n|\mathcal{F}_{n-1}\right) \\
&= E\left(S_n^2 - S_{n-1}^2 + S_{n-1}^2|\mathcal{F}_{n-1}\right) - \sigma^2 n \\
&= E\left(S_n^2 - S_{n-1}^2|\mathcal{F}_{n-1}\right) - \sigma^2 + \left[E\left(S_{n-1}^2|\mathcal{F}_{n-1}\right) - (n-1)\,\sigma^2\right] \\
&= E\left(X_n^2 + 2S_{n-1}X_n|\mathcal{F}_{n-1}\right) - \sigma^2 + M_{n-1} \\
&= E\left(X_n^2|\mathcal{F}_{n-1}\right) - \sigma^2 + 2E\left(S_{n-1}X_n|\mathcal{F}_{n-1}\right) + M_{n-1} \\
&= E\left(X_n^2\right) - \sigma^2 + 2E\left(X_n\right)S_{n-1} + M_{n-1} \\
&= M_{n-1}
\end{aligned}$$

Using the hint from the script ($S_n^2 - S_{n-1}^2 = X_n^2 + 2S_{n-1}X_n$). Knowing that S_{n-1} is $\mathcal{F}_{n-1}$-measurable and that X_n is independent of $\mathcal{F}_{n-1}$ (as the X_n are independent) we were able to use the Law of redundant conditioning.

Intermediate 19 (Test Feb-05), *8.5 (Notes Feb-01)* ✓

Problem: Show that $W_t/t \xrightarrow{P} 0$ as $t \to \infty$.

——

Show that : $\frac{W_t}{t} \to^P 0$ as t $\to \infty$

Proof: we need to show that $P(|\frac{w_t}{t}| > \epsilon) \to 0, \forall \epsilon > 0, ast \to \infty$

Apply Chebeysev $p(|\frac{w_t}{t}| > \epsilon) \leq \frac{1}{\epsilon^2}\int \frac{w_t^2}{t^2}dP$ (1)

Since $W_t \sim \mathcal{N}(0,t)$ we must have $E(W_t) = 0, Var(W_t) = t$

Consequently $E(\frac{W_t}{t}) = 0, var(\frac{W_t}{t}) = E(\frac{W_t^2}{t^2}) - (E(\frac{W_t}{t}))^2 = \frac{1}{t^2}t = \frac{1}{t}$

Hence $\mathrm{E}(\frac{W_t^2}{t^2}) = \frac{1}{t}, i.e \int \frac{W_t^2}{t^2} dP = \frac{1}{t}$

Pluggin $\int \frac{W_t^2}{t^2} dP = \frac{1}{t}$ into (1), we have $P(|\frac{W_t}{t}| > \epsilon) \leq \frac{1}{\epsilon^2}\frac{1}{t} \to 0$, as t $\to \infty$ $\forall \epsilon > 0$

Intermediate 20 (Test Feb-05), *8.12 (Notes Feb-01)* ✓

Problem: Let $(W_t)_{t\geq 0}$ be a Wiener process. For every $t > 0$ and every Riemannian sequence of subdivisions $0 = t_0^n < t_1^n < \ldots < t_n^n = t$

$$\sum_{i=1}^{n} |W(t_i^n) - W(t_{i-1}^n)|^2 \xrightarrow{P} t, \quad t > 0.$$

We define $Q_n := \sum_{i=1}^n (W_{t_i} - W_{t_{i-1}})^2$. (W_t) is a Wiener process, hence we know that $\mathrm{E}\left[(W_{t_i} - W_{t_{i-1}})^2\right] = \mathrm{Var}[W_{t_i} - W_{t_{i-1}}] = t_i - t_{i-1}$ and therefore

$$\mathrm{E}[Q_n] = \sum_{i=1}^{n} \mathrm{E}\left[(W_{t_i} - W_{t_{i-1}})^2\right] = \sum_{i=1}^{n} t_i - t_{i-1} = t\,.$$

Now we compute $\mathrm{Var}(Q_n)$. If X is normally distributed $\mathrm{E}[X^4] = 3(\mathrm{Var}[X])^2$. Thus we get:

$$\begin{aligned}\mathrm{Var}(Q_n) &= \sum_{i=1}^{n} \mathrm{Var}((W_{t_i} - W_{t_{i-1}})^2) \leq \sum_{i=1}^{n} 3(t_i - t_{i-1})^2 \\ &\leq \max_i (t_i - t_{i-1}) \sum_{i=1}^{n} 3(t_i - t_{i-1}) = 3t \max_i (t_i - t_{i-1}) \to 0\end{aligned}$$

Now we consider Chebyshev's inequality, which yields:

$$P(|Q_n - 1| > \varepsilon) \leq \frac{1}{\varepsilon^2}\mathrm{Var}[Q_n] \to 0$$

Intermediate 21 (Test Feb-05), *8.27 (Notes Feb-01)* ✓

Problem: Let $(X_t)_{t\geq 0}$ be an integrable process adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$. Show that (a) implies (b):

(a) $E(X_\sigma) = E(X_0)$ for all bounded stopping times σ.

(b) $(X_t)_{t\geq 0}$ is a martingale.

We have to show $\mathrm{E}[X_t|\mathcal{F}_s] = X_s$ for all $s < t < \infty$, ie. $\int_F X_t\,dP = \int_F X_s\,dP$ for all $F \in \mathcal{F}_s$.

Fix a arbitrary but fixed $F \in \mathcal{F}_s$. Define $\tau := s\,1_F + t\,1_{F^C}$. τ is bounded by t and is a stopping time and is a stopping time, because

$$(\tau \leq r) = \begin{cases} \emptyset & \text{when } r < s \\ F & \text{when } r \geq s \text{ and } r < t \\ \Omega & \text{when } r \geq t \end{cases}$$

where $\emptyset, \Omega \in \mathcal{F}_r$ and $F \in \mathcal{F}_s \subseteq \mathcal{F}_r$ for $r \geq s$. Therefore we know by assumption $\mathrm{E}[X_\tau] = \mathrm{E}[X_0]$.

By the same argument we get $\tilde{\tau} := t\,1_\Omega$ is a bounded stopping time and $\mathrm{E}[X_t] = \mathrm{E}[X_{\tilde{\tau}}] = \mathrm{E}[X_0]$. Together we have

$$\mathrm{E}[X_t] = \mathrm{E}[X_\tau]$$

and therefore

$$\int_F X_t\,dP + \int_{F^C} X_t\,dP = \int_F X_s\,dP + \int_{F^C} X_t\,dP\,,$$

from which we get the result:

$$\int_F X_t\,dP = \int_F X_s\,dP$$

Intermediate 22 (Test Feb-05), *8.32a (Notes Feb-01)* ✓

PROBLEM: Let $(W_t)_{t\geq 0}$ be a Wiener process and let $\tau_{a,b}$ be the first passage time of the boundary $f(t) = a + bt$, $a > 0$. Take for granted that

$$E(e^{-\lambda\tau_{a,b}}1_{(\tau_{a,b}<\infty)}) = e^{-a(b+\sqrt{b^2+2\lambda})},\ \lambda \geq 0$$

Show that $P(\tau_{0,b} = 0) = 1$ for every $b > 0$.

——

The proof is provided by Prof. Strasser in the last version of the midterm test problems file. Here are the answers to the two questions posed within his solution:

Q1) Which limit theorem for integrals can be applied?
We apply the dominated convergence theorem (theorem 3.31 (script version 2006-02-05)).

Q2) How does the assertion follow from the identity?

Claim:

$$E(e^{-\lambda\tau_{0,b}}1_{(\tau_{0,b}<\infty)}) = 1\ \Rightarrow\ P(\tau_{0,b} = 0) = 1$$

Proof: Note that

$$e^{-\lambda\tau_{0,b}}1_{(\tau_{0,b}<\infty)} \leq 1(**)$$

$$\Leftrightarrow e^{-\lambda\tau_{0,b}}1_{(\tau_{0,b}<\infty)} \leq E(e^{-\lambda\tau_{0,b}}1_{(\tau_{0,b}<\infty)})$$

(*)
Taking expectations on both sides of (*) we receive

$$E(e^{-\lambda\tau_{0,b}}1_{(\tau_{0,b}<\infty)}) \leq E(e^{-\lambda\tau_{0,b}}1_{(\tau_{0,b}<\infty)})$$

which must hold with equality. Thus all the above inequalities must hold with equality. But (**) can be an equality only if $\tau_{0,b} = 0$. QED

Intermediate 23 (Test Feb-05), *8.37 (Notes Feb-01)* ✓

PROBLEM: Let $(W_t)_{t\geq 0}$ be a Wiener process, $c, d > 0$ and define

$$\sigma_{c,d} = \inf\{t : W_t \notin (-c, d)\}$$

Find the distribution of $W_{\sigma_{c,d}}$.

——

Proof by Prof. Strasser: We may assume (knowing from a previous problem) that $\sigma_{c,d}$ is a stopping time and $\sigma_{c,d} < \infty$ P-a.s.

Applying the optimal stopping theorem to the truncated stopping times $\sigma_{c,d} \cap n$ we obtain that

$$E\left(W_{\sigma_{c,d}\cap n}\right) = 0$$

The random variables $W_{\sigma_{c,d}\cap n}$ have values in the interval $[-c, d]$ and thus are uniformly bounded, hence dominated. With $n \to \infty$ we may apply Lebesgues theorem to obtain

$$E\left(W_{\sigma_{c,d}}\right) = 0$$

which means

$$-cP\left(W_{\sigma_{c,d}} = -c\right) + dP\left(W_{\sigma_{c,d}} = d\right) = 0$$

Since $P\left(W_{\sigma_{c,d}} = -c\right) + P\left(W_{\sigma_{c,d}} = d\right) = 1$, the distribution can be found easily:

$$P\left(W_{\sigma_{c,d}} = -c\right) + P\left(W_{\sigma_{c,d}} = d\right) = 1$$

$$P\left(W_{\sigma_{c,d}} = d\right) = 1 - P\left(W_{\sigma_{c,d}} = -c\right)$$

Substituting, we obtain

$$-cP\left(W_{\sigma_{c,d}} = -c\right) + d\left(1 - P\left(W_{\sigma_{c,d}} = -c\right)\right) = 0$$

$$(c + d)\ P\left(W_{\sigma_{c,d}} = -c\right) = d$$

$$P\left(W_{\sigma_{c,d}} = -c\right) = \frac{d}{c + d}$$

similarly:

$$P\left(W_{\sigma_{c,d}} = d\right) = \frac{c}{c+d}$$

Intermediate 24 (Test Feb-05), *8.38 (Notes Feb-01)* ✓

PROBLEM: Let $(W_t)_{t\geq 0}$ be a Wiener process, $c, d > 0$ and define

$$\sigma_{c,d} = \inf\{t : W_t \notin (-c, d)\}$$

Find $E(\sigma_{c,d})$.

——

Let $(W_t)_{t\geq 0}$ be a Wiener process, $c, d > 0$ and define

$$\sigma_{c,d} = \inf\{t : W_t \notin (-c, d)\}$$

Find $E(\sigma_{c,d})$.

Proof:

We may assume that $\sigma_{c,d}$ is a stopping time and that $\sigma_{c,d} < \infty$ P-a.s. Applying the optimal stopping theorem to the truncated stopping times $\sigma_{c,d} \cap n$ we obtain that

$$E(W^2_{\sigma_{c,d\cap n}}) = E(\sigma_{c,d\cap n})$$

The random variables $W_{\sigma_{c,d\cap n}}$ have values in the interval $[-c, d]$ and thus are uniformly bounded and hence dominated. With $n \to \infty$ we may apply Lebesgue's Theorem on the left hand side and Beppo Levi's Theorem on the right hand side to obtain:

$$E(W^2_{\sigma_{c,d}}) = E(\sigma_{c,d})$$

This gives $E(\sigma_{c,d}) = c^2P(W_{\sigma_{c,d}} = -c) + d^2P(W_{\sigma_{c,d}} = d)$ From the definition of the Wiener Process we know that $E(W_{\sigma_{c,d}}) = 0$ which means that $cP(W_{\sigma_{c,d}} = -c) + dP(W_{\sigma_{c,d}} = d) = 0$ Since $P(W_{\sigma_{c,d}} = -c) + P(W_{\sigma_{c,d}} = d) = 1$, we obtain

1. $P(W_{\sigma_{c,d}} = -c) = \frac{d}{c+d}$

2. $P(W_{\sigma_{c,d}} = d) = \frac{c}{c+d}$

Substitute (1) and (2) into $E(\sigma_{c,d}) = c^2 P(W_{\sigma_{c,d}} = -c) + d^2 P(W_{\sigma_{c,d}} = d)$, which gives us finally

$$E(\sigma_{c,d}) = \frac{cd^2}{c+d} + \frac{cd^2}{c+d} = cd$$

3 Advanced Questions

Advanced 1 (Test Feb-05), *1.10 (Notes Feb-01)* ✓

Problem: (a) Show that any content λ_α defined by

$$\lambda_\alpha((a,b]) := \alpha(b) - \alpha(a) \tag{5}$$

necessarily satisfies

$$A = \bigcup_{i=1}^{n} I_i, \text{ where } (I_i) \text{ are pw. dj. intervals } \Rightarrow \lambda_\alpha(A) = \sum_{i=1}^{n} \lambda_\alpha(I_i) \tag{6}$$

(b) Show that using (6) as a definition is unambiguous.

(c) Show that (6) defines a content on $\mathcal{R}$ which is finite on bounded sets.

(a) If λ_α is a content, then it is additive, ie. for $A = \bigcup_{i=1}^{n}(I_i)$, where I_i p.w. disjoint intervals, we get $\lambda_\alpha(A) = \sum_{i=1}^{n} \lambda_\alpha(I_i)$.

(b) Note $\mathcal{R}$ is a field. Let $\mathcal{G} := \{(a,b] : a,b \in \mathbf{R}, a < b\}$. Let $A \in \mathcal{R}$ and $A = \bigcup_{i=1}^{m} A_i = \bigcup_{j=1}^{n} B_j$ where A_i are p.w. disjoint intervals and B_i too. We want to show that $\sum_{i=1}^{n} \lambda_\alpha(A_i) = \sum_{j=1}^{n} \lambda_\alpha(B_j)$.

- First we consider the case $m = 1$, ie. $(a,b] = \bigcup_{j=1}^{n}(a_j, b_j]$. W.l.o.g. $a = a_1 \leq b_1 = a_2 \leq b_2 = \cdots \leq b_{n-1} = a_n \leq b_n = b$. But then we have

$$\sum_{j=1}^{n} \lambda_\alpha((a_j, b_j]) = \sum_{j=1}^{n} \alpha(b_j) - \alpha(a_j) = \alpha(b) - \alpha(a)\,.$$

- Now we have the general case $m \geq 1$. The B_j are disjoint, hence $A_i = \bigcup_{j=1}^{n} A_i \cap B_j$ and analogously $B_j = \bigcup_{i=1}^{m} A_i \cap B_j$. Since $A_i \cap B_j \in \mathcal{G}$ we get by using part 1):

$$\sum_{i=1}^{m} \lambda_\alpha(A_i) = \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_\alpha(A_i \cap B_j) = \sum_{j=1}^{n}\sum_{i=1}^{m} \lambda_\alpha(A_i \cap B_j) = \sum_{j=1}^{n} \lambda_\alpha(B_j)$$

(c) Assume $A \in \mathcal{R}$ bounded, ie. $A \subseteq (-a, a]$ for some $a \in \mathrm{R}$, but then

$$\lambda_\alpha(A) \leq \lambda_\alpha((-a, a]) = 2a$$

Advanced 2 (Test Feb-05), *1.28 (Notes Feb-01)* ✓

PROBLEM: (a) The system 2^Ω (system of all subsets of Ω is a σ-field. (b) The intersection of any family of σ-fields is a σ-field. (c) Let $\mathcal{C}$ be any system of subsets on Ω and denote by $\sigma(\mathcal{C})$ the intersection of all σ-fields containing $\mathcal{C}$:

$$\sigma(\mathcal{C}) = \bigcap_{\mathcal{C} \subseteq \mathcal{F}} \mathcal{F}$$

Then $\sigma(\mathcal{C})$ is the smallest σ-field containing $\mathcal{C}$:

$$\mathcal{C} \subseteq \mathcal{F},\ \mathcal{F} \text{ is a } \sigma\text{-field} \ \Rightarrow\ \mathcal{C} \subseteq \sigma(\mathcal{C}) \subseteq \mathcal{F}$$

(a) 1) $\Omega \subseteq \Omega$

2) $A_1, A_2 \subseteq \Omega$ then $A_1 \cup A_2 \subseteq \Omega$

3) $A \subseteq \Omega$ then $A^C = \Omega \setminus A \subseteq \Omega$

4) $(A_i)_{i \in \mathbf{N}}$, $A_i \subseteq \Omega$ then $\bigcup_{i \in \mathbf{N}} A_i \subseteq \Omega$

(b) $(\mathcal{A}_\alpha)_{\alpha \in A}$, $\mathcal{A}_\alpha$ is a σ-field on Ω for all $\alpha \in A$. Define $\mathcal{A} := \bigcap_{\alpha \in A} \mathcal{A}_\alpha$. We show $\mathcal{A}$ is a σ-field.

1) $\Omega \in \mathcal{A}_\alpha$ for all $\alpha \in A$, hence $\Omega \in \bigcap_{\alpha \in A} \mathcal{A}_\alpha = \mathcal{A}$.

2) $A_1, A_2 \in \mathcal{A}$, ie. $A_1 \cup A_2 \in \mathcal{A}_\alpha$ for all $\alpha \in A$, because $\mathcal{A}_\alpha$ is a σ-field. Hence $A_1 \cup A_2 \in \bigcap_{\alpha \in A} \mathcal{A}_\alpha = \mathcal{A}$.

3) $A \in \mathcal{A}$, ie. $A^C \in \mathcal{A}_\alpha$ for all $\alpha \in A$, because $\mathcal{A}_\alpha$ is a σ-field. Hence $A^C \in \bigcap_{\alpha \in A} \mathcal{A}_\alpha = \mathcal{A}$.

4) Consider $(A_i)_{i\in\mathbf{N}}$, $A_i \in \mathcal{A}$ for all $i \in \mathbf{N}$, ie. $A_i \in \mathcal{A}_\alpha$ for all $i \in \mathbf{N}$ and $\bigcup_{i\in\mathbf{N}} A_i \in \mathcal{A}_\alpha$ for all $\alpha \in A$, because $\mathcal{A}_\alpha$ is a σ-field. Hence $\bigcup_{i\in\mathbf{N}} A_i \in \bigcap_{\alpha\in A} \mathcal{A}_\alpha = \mathcal{A}$.

(c) By (b) we know that $\sigma(\mathcal{C})$ is a σ-field. It is the smallest one by definition: Let $\tilde{\mathcal{F}}$ be a σ-field containing $\mathcal{C}$, then

$$\sigma(\mathcal{C}) = \bigcap_{\mathcal{C}\subseteq\mathcal{F}} \mathcal{F} = \tilde{\mathcal{F}} \cap \left(\bigcap_{\mathcal{C}\subseteq\mathcal{F}\neq\tilde{\mathcal{F}}} \mathcal{F} \right) \subseteq \tilde{\mathcal{F}}\,.$$

Advanced 3 (Test Feb-05), *1.40a (Notes Feb-01)* ✓

Problem: Assume that $\mu(\Omega) < \infty$. Show that for every $M \in \mathcal{M}$ and every (arbitrarily small) $\epsilon > 0$ there is a set $A \in \mathcal{A}$ such that $\mu(M \setminus A) < \epsilon$ and $\mu(A \setminus M) < \epsilon$.

First consider the case that μ is a finite measure on Ω. Let $\varepsilon > 0$ and $M \in \mathcal{M}$ be arbitrary but fixed. By looking at the definition of μ^* we know there exist $A_i \subseteq \mathcal{A}, M \subseteq \bigcup_{i\in\mathbf{N}} A_i$ such that

$$\sum_{i\in\mathbf{N}} \mu(A_i) - \mu(M) < \varepsilon\,.$$

Now define $B_n := \bigcup_{i=1}^{n} A_i$ and $B := \bigcup_{i\in\mathbf{N}} A_i$. By definition $B_n \uparrow B$ and $B \setminus B_n \downarrow \emptyset$ respectively, therefore by Lemma 1.6 $\mu(B \setminus B_n) \downarrow 0$, ie. there exists an $N \in \mathbf{N}$ such that $\mu(B \setminus B_N) < \varepsilon$. Define $A := B_N \in \mathcal{A}$. We have constructed the sets B and A so that the following is true: $(M \setminus A) \subseteq (B \setminus A)$ and $(A \setminus M) \subseteq (B \setminus M)$. Thus we get

$$\mu(M \setminus A) \leq \mu(B \setminus A) < \varepsilon$$

and

$$\mu(A \setminus M) \leq \mu(B \setminus M) \leq \sum_{i\in\mathbf{N}} \mu(A_i) - \mu(M) < \varepsilon\,.$$

Now we allow for μ being σ-finite, ie. w.l.o.g. there exists a sequence $\Omega_i \in \sigma(\mathcal{A})$, pairwise disjoint, such that $\Omega = \bigcup_{i\in\mathbf{N}} \Omega_i$ and $\mu(\Omega_i) < \infty$ for all $i \in \mathbf{N}$ (if the original sequence Ω_i was not disjoint, consider $\tilde{\Omega}_1 := \Omega_1$ and $\tilde{\Omega}_i := \Omega_i \setminus \left(\bigcup_{j=1}^{i-1} \tilde{\Omega}_j\right)$). Let $\varepsilon > 0$ and $M \in \mathcal{M}$ be arbitrary but fixed. Define $M_i := M \cap \Omega_i$ for all $i \in \mathbf{N}$. By the first part we know that there exist sets $C_i \subseteq \Omega_i$ such that $\mu(M_i \setminus C_i) < 2^{-i}\,\varepsilon$ and $\mu(C_i \setminus M_i) < 2^{-i}\,\varepsilon$. By construction the sets M_i are pairwise disjoint. This is also true for the C_i. Hence we get for $A := \bigcup_{i\in\mathbf{N}} C_i$ (remember $M = \bigcup_{i\in\mathbf{N}} M_i$)

$$\begin{aligned}\mu(M \setminus A) &= \mu(\bigcup_{i\in\mathbf{N}} M_i \setminus \bigcup_{i\in\mathbf{N}} C_i) = \mu(\bigcup_{i\in\mathbf{N}} (M_i \setminus C_i)) \\ &= \sum_{i\in\mathbf{N}} \mu(M_i \setminus C_i) < \sum_{i\in\mathbf{N}} 2^{-i}\,\varepsilon = \epsilon\end{aligned}$$

and

$$\begin{aligned}\mu(A \setminus M) &= \mu(\bigcup_{i\in\mathbf{N}} C_i \setminus \bigcup_{i\in\mathbf{N}} M_i) = \mu(\bigcup_{i\in\mathbf{N}} (C_i \setminus M_i)) \\ &= \sum_{i\in\mathbf{N}} \mu(C_i \setminus M_i) < \sum_{i\in\mathbf{N}} 2^{-i}\,\varepsilon = \epsilon\,.\end{aligned}$$

Advanced 4 (Test Feb-05), *1.40b (Notes Feb-01)* ✓

> Problem: Assume that $\mu(\Omega) < \infty$. Show that for every $M \in \mathcal{M}$ there is some $A \in \sigma(\mathcal{A})$ such that $\mu(M \setminus A) = 0$ and $\mu(A \setminus M) = 0$.

Again, first consider the case that μ is a finite measure on Ω. We use the result from the first part of the exercise, ie. for every $m \in \mathbf{N}$ there exists an $A_m \in \mathcal{A}$ such that $\mu(M \setminus A_m) < 2^{-m}$ and $\mu(A_m \setminus M) < 2^{-m}$. Define $B_n := \bigcup_{m=n}^{\infty} A_m \in \sigma(\mathcal{A})$ and $A := \bigcap_{n=1}^{\infty} B_n \in \sigma(\mathcal{A})$. We know that

$$\begin{aligned}&\mu(M \setminus B_n) \le \mu(M \setminus A_n) < 2^{-n} \quad \text{and} \\ &\mu(B_n \setminus M) \le \sum_{i=1}^{n} \mu(A_i \setminus M) < 2^{-n} \sum_{i=0}^{n} 2^{-i} = 2^{-n+1}\,.\end{aligned}$$

Hence $\mu(M \setminus B_n) \to 0$ and $\mu(B_n \setminus M) \to 0$. But we know that $B_n \downarrow A$, hence $(M \setminus B_n) \uparrow (M \setminus A)$ and $(B_n \setminus M) \downarrow (A \setminus M)$. By Lemma 1.6(a) and 1.6(c) we get $\mu(M \setminus B_n) \uparrow \mu(M \setminus A)$ and $\mu(B_n \setminus M) \downarrow \mu(A \setminus M)$. Plugging everything together we get

$$\mu(M \setminus A) = \mu(A \setminus M) = 0\,.$$

Now let μ be σ-finite. Take the sequence Ω_i from above. Again define $M_i := M \cap \Omega_i$. By the first part we know that there exist sets $D_i \subseteq \Omega_i$ such that $\mu(M_i \setminus D_i) = \mu(D_i \setminus M_i) = 0$. Define $A := \bigcup_{i\in\mathbf{N}} D_i$ again and consider the same arguments from exercise 1.36 (1):

$$\mu(M \setminus A) = \mu(\bigcup_{i\in\mathbf{N}} M_i \setminus \bigcup_{i\in\mathbf{N}} D_i) = \mu(\bigcup_{i\in\mathbf{N}} (M_i \setminus D_i)) = \sum_{i\in\mathbf{N}} \mu(M_i \setminus D_i) = 0$$
$$\mu(A \setminus M) = \mu(\bigcup_{i\in\mathbf{N}} D_i \setminus \bigcup_{i\in\mathbf{N}} M_i) = \mu(\bigcup_{i\in\mathbf{N}} (D_i \setminus M_i)) = \sum_{i\in\mathbf{N}} \mu(D_i \setminus M_i) = 0$$

Advanced 5 (Test Feb-05), *2.9 (Notes Feb-01)* ✓

> PROBLEM: Let $f : (\Omega, \mathcal{A}) \to (Y, \mathcal{B})$ and let $\mathcal{C}$ be a generating system of $\mathcal{B}$, i.e. $\mathcal{B} = \sigma(\mathcal{C})$. Then f is $(\mathcal{A}, \mathcal{B})$-measurable iff $f^{-1}(C) \in \mathcal{A}$ for all $C \in \mathcal{C}$.

---

f is $(\mathcal{A}, \mathcal{B})$-measurable implies $f^{-1}(C) \in \mathcal{A}$ for all $C \in \mathcal{C}$ is trivial to show, because consider some $C \in \mathcal{C} \subseteq \mathcal{B} = \sigma(\mathcal{C})$. But f is $(\mathcal{A}, \mathcal{B})$-measurable, thus we know that $f^{-1}(C) \in \mathcal{A}$.

Now we show that the converse is true too. Define $\mathcal{D} := \{D \subseteq Y : f^{-1}(D) \in \mathcal{A}\}$. We prove that $\mathcal{D}$ is a σ-field:

1. If $D_1, D_2 \in \mathcal{D}$, ie. $f^{-1}(D_1), f^{-1}(D_2) \in \mathcal{A}$ then

$$f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2) \in \mathcal{A}\,.$$

2. Let $Y = Y_1 \cup Y_2$, where $f^{-1}(Y_1) = \Omega$ and $f^{-1}(Y_2) = \emptyset$. Then

$$f^{-1}(Y) = f^{-1}(Y_1 \cup Y_2) = f^{-1}(Y_1) \cup f^{-1}(Y_2) = \Omega \cup \emptyset = \Omega\,.$$

   Thus $f^{-1}(Y) = \Omega \in \mathcal{A}$ and $Y \in \mathcal{D}$.

3. Consider $D \in \mathcal{D}$, ie. $f^{-1}(D) \in \mathcal{A}$. Then

$$f^{-1}(D^C) = f^{-1}(Y \setminus D) = f^{-1}(Y) \setminus f^{-1}(D) = (f^{-1}(D))^C\,.$$

4. Suppose $\{D_i\}_{i\in\mathbf{N}}$ a sequence of pairwise disjoint sets $D_i \in \mathcal{D}$. By definition $f^{-1}(D_i) \in \mathcal{A}$ and because $\mathcal{A}$ is a σ-field:

$$f^{-1}(\bigcup_{i\in\mathbf{N}} D_i) = \bigcup_{i\in\mathbf{N}} f^{-1}(D_i) \in \mathcal{A}$$

   But this is definition of $\bigcup_{i\in\mathbf{N}} D_i \in \mathcal{D}$.

By assumption we know that $\mathcal{C} \subseteq \mathcal{D}$. But $\mathcal{D}$ is a sigma-field, so it definitely contains $\sigma(\mathcal{C})$, ie. $\mathcal{B} = \sigma(\mathcal{C}) \subseteq \mathcal{D}$. By the definition of $\mathcal{D}$ we now know that f is $(\mathcal{A}, \mathcal{B})$-measurable.

Advanced 6 (Test Feb-05), *3.12 (Notes Feb-01)* ✓

Problem: Prove Fatou's lemma: For every sequence (f_n) of nonnegative measurable functions

$$\liminf_n \int f_n\,d\mu \geq \int \liminf_n f_n\,d\mu$$

Define $g_k := \inf_{n\geq k} f_n$. Then g_k is increasing (and non-negative) by definition. We know that $g_k \leq f_n$ for all $n \geq k$ and therefore for all $k \in \mathbf{N}$

$$\int g_k\,d\mu \leq \inf_{n\geq k} \int f_n\,d\mu\,.$$

Now we apply Beppo-Levi for the sequence g_k and get

$$\int \liminf_{n\to\infty} f_n \, d\mu = \int \lim_{k\to\infty} \inf_{n\geq k} f_n \, d\mu = \int \lim_{k\to\infty} g_k \, d\mu = \lim_{k\to\infty} \int g_k \, d\mu$$
$$\leq \lim_{k\to\infty} \inf_{n\geq k} \int f_n \, d\mu = \liminf_{n\to\infty} \int f_n \, d\mu \, .$$

Advanced 7 (Test Feb-05), *7.6 (Notes Feb-01)* ✓

PROBLEM: Let (S_n) be a random walk on $\mathbb{Z}$. Show that

$$P(\sup_n S_n \geq b) = \begin{cases} 1 & \text{whenever } p \geq 1/2 \\ \left(\dfrac{p}{1-p}\right)^b & \text{whenever } p < 1/2 \end{cases}$$

(Take the formulas for $q_0(a)$ and $q_c(a)$ for granted.)

——

i

show $q_{a+b}(a) \to P(\sup_n S_n \geq b)$ as $a \to \infty$

$$\begin{aligned} M_a &:= \{\exists n \in N : S_k > -a \quad \forall k \leq n-1, S_n = b\} \\ M &:= \{\exists n \in N : S_n = b\}\,, \qquad S_0 := 0 \end{aligned}$$

obviously $M_a \uparrow M$ as $a \to \infty$ since $\{S_k > -a\} \uparrow \Omega$ as $a \to \infty$.

$$\begin{aligned} q_{a+b}(a) &= P\left(\tau_{a+b} < \tau_0 | a + S_0 = a\right) \\ &= P\left(\exists n \in N : a + \sum_{i=1}^{k} X_i > 0 \quad \forall\, k \leq n-1, a + \sum_{i=1}^{n} X_i = a + b\right) \\ &= P\left(\exists n \in N : S_k > -a \quad \forall\, k \leq n-1, S_n = b\right) \\ &= P(M_a) \end{aligned}$$

By Lemma 1.7 (Notes 5/2) $P(M_a) \uparrow P(M)$ as $M_a \uparrow M$ (this happens when $a \to \infty$). $P(M) = P(\exists n \in N : S_n = b) = P(sup_n S_n \geq b)$

ii

$$\text{Show } q_{a+b}(a) \to \begin{cases} 1 & p \geq 1/2 \\ \left(\frac{p}{1-p}\right)^b & p < 1/2 \end{cases}$$

From Discussion 7.2 in the notes (5/2) we know that

$$q_{a+b}(a) \to \begin{cases} \frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^{a+b} - 1} & p \neq 1/2 \\ \frac{a}{a+b} & p = 1/2 \end{cases}$$

case 1: p > 1/2 $\frac{1-p}{p} < 1 \Rightarrow \left(\frac{1-p}{p}\right)^a \to 0$ as $a \to \infty$

$\Rightarrow q_{a+b}(a) = \frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^{a+b} - 1} \to \frac{0-1}{0-1} = 1$ as $a \to \infty$

case 2: p = 1/2 $q_{a+b}(a) = \frac{a}{a+b} \to 1$ as $a \to \infty$

case 3: p < 1/2 $\frac{1-p}{p} > 1 \Rightarrow \left(\frac{1-p}{p}\right)^{-a} \to 0$ as $a \to \infty$

$\Rightarrow q_{a+b}(a) = \frac{1 - \left(\frac{1-p}{p}\right)^{-a}}{\left(\frac{1-p}{p}\right)^b - \left(\frac{1-p}{p}\right)^{-a}} \to \frac{1-0}{\left(\frac{1-p}{p}\right)^b} = \left(\frac{p}{1-p}\right)^b$ as $a \to \infty$

Combining (i) and (ii), the proof is done.

Advanced 9 (Test Feb-05), *8.10 (Notes Feb-01)* ✓

Problem: Show that the quadratic variation of a continuous BV-function is zero on every compact interval.

Let $a = t_0^n < \cdots < t_n^n = b$ a Riemannian sequence of subdivisions of $[a, b]$. Then

$$\begin{aligned}
0 &\le \lim_{n\to\infty} \sum_{i=1}^{n} |f(t_i^n) - f(t_{i-1}^n)|^2 \\
&\le \lim_{n\to\infty} \left[\max_{r=1,\dots,n} |f(t_r^n) - f(t_{r-1}^n)| \; \sum_{i=1}^{n} |f(t_i^n) - f(t_{i-1}^n)| \right] \\
&\le \lim_{n\to\infty} \left[\max_{r=1,\dots,n} |f(t_r^n) - f(t_{r-1}^n)| \; \left(\lim_{m\to\infty} \sum_{i=1}^{m} |f(t_i^m) - f(t_{i-1}^m)| \right) \right] \\
&= V_a^b(f) \; \lim_{n\to\infty} \left[\max_{r=1,\dots,n} |f(t_r^n) - f(t_{r-1}^n)| \right]
\end{aligned}$$

but $\max_{r=1,\dots,n} |f(t_r^n) - f(t_{r-1}^n)| \to 0$ as $|t_r^n - t_{r-1}^n| \to 0$, since f is continous on a compact interval. This shows

$$\lim_{n\to\infty} \sum_{i=1}^{n} |f(t_i^n) - f(t_{i-1}^n)|^2 \to 0\,.$$

4 Review Questions

Review 1 (Test Feb-05), *1.19 (Notes Feb-01)*

PROBLEM: Explain the structure and generation of finite fields. How to define contents on finite fields ?

—

A finite field consists of finitely many subsets. Let $\mathcal{C} = (C_1, C_2, ..., C_m)$ be the vector of finitely many subsets with the following properties:

1. The subsets are pairwise disjoint.
2. The subsets are exhaustive: $C_1 \cup C_2 \cup ... \cup C_m = \Omega$

We say that $\mathcal{C}$ is a finite partition of Ω. The partition $\mathcal{C}$ generates the field $\mathcal{R} \subseteq \Omega$ when the following equation holds

$$\mathcal{R} := \left\{ \bigcup_{i \in \alpha} C_i : \alpha \subseteq (1, ..., m) \right\}.$$

Then $\mathcal{R}$ is the smallest field containing $\mathcal{C}$. Every finite field is generated by a partition. A content on $\mathcal{R}$ is defined in the following way: for any number $a_i \geq 0, \mu(C) := \sum_{C_i \subseteq C} a_i$ defines a content on $\mathcal{R}$.

Review 2 (Test Feb-05), *1.36 (Notes Feb-01)*

PROBLEM: What is a field and what is a σ-field ? What is the difference between a content and a measure.

—

What is a field and what is a σ-field?

A field on a set $\Omega \neq \emptyset$ is a system $\mathcal{A}$ of subsets $A \subseteq \Omega$ which satisfies the following conditions:

1. $\Omega \in \mathcal{A}$, $\emptyset \in \mathcal{A}$
2. $A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \cup A_2 \in \mathcal{A}$ and $A_1 \cap A_2 \in \mathcal{A}$
3. $A \in \mathcal{A} \Rightarrow A^C \in \mathcal{A}$.

$\mathcal{A}$ is a σ-field if in addition to (1)-(3):

4. For every sequence $(A_i)_{i \in N} \in \mathcal{A} \Rightarrow \bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{A}$ and $\bigcap\limits_{i=1}^{\infty} A_i \in \mathcal{A}$

What is the difference between a content and a measure?

A content is a set function μ defined on a field $\mathcal{A}$ such that

1. $\mu(A) \in [0, \infty]$ whenever $A \in \mathcal{A}$
2. $\mu(\emptyset) = 0$
3. Finite additivity: $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$ whenever $A_1, A_2 \in \mathcal{A}$ and $A_1 \cap A_2 = \emptyset$, more formally whenever $\mu\left(\bigcup\limits_{i=1}^{n} A_i\right) = \sum\limits_{i=1}^{n} \mu(A_i)$ $\forall$ finite pairwise disjoint sets $A_1 ... A_n \in \mathcal{A}$

If we choose instead of (3) property (4), we get the definition of a measure:

4. σ-additivity: $\mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \sum\limits_{n=1}^{\infty} \mu(A_n)$ $\forall$ finite pairwise disjoint sets $A_1 ... A_n \in \mathcal{A}$

$\Rightarrow$A measure is a σ-additive content defined on a σ-field.

Review 3 (Test Feb-05), *1.37 (Notes Feb-01)*

Problem: Explain the ideas of generating a σ-field by a system of sets. Explain how the measurable spaces $(\mathbb{N}, 2^{\mathbb{N}})$, $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ are generated.

——

Note that a σ-field is but a system of sets (a set of sets) that satisfies the three characteristic properties (see review question 2). An arbitrary set of sets, let's denote it by C, may not display all (or any) of these properties. However, if it contained some more sets, it would. The idea, therefore, is "to augment" that set (of sets) C with further sets, so that it does become a σ-field. Note that there is not one unique such larger set of sets containing C (larger meaning with higher cardinality), since one can possibly add more sets than minimally necessary. [Say, $\{\emptyset\}$ is no σ-field (because it does not contain omega which must not be the empty set), but both $\{\emptyset, \Omega\}$ and $\{\emptyset, A, A^c, \Omega\}$ contain the empty set and are, for every $A \in \Omega$ and any Ω.] In this spirit we define $\sigma(C)$, the σ-field generated by C, such that it fulfills
a) $C \subseteq \sigma(C)$
b) $\forall$ σ-fields S where $C \subseteq S : \sigma(C) \subseteq S$
so that we obtain the minimal sigma-field containing the set C, which is called the generator of the σ-field.
Note that the following is always true:
o) $\sigma(C)=\bigcap_{S_i \in \Sigma} S_i$ where Σ is the set of all σ-fields containing C
o) for any σ-field S: $C \subseteq S \Rightarrow C \subseteq \sigma(C) \subseteq S$

Generating $(\mathbf{N},2^{\mathbf{N}})$:

The set of all singleton sets containing each natural number is, for obvious reasons (e.g., it does not contain complements), no σ-field on the natural numbers. However, in view of what has been established, it generates one. The generated σ-field is the power set, which is generally true for one-point sets on countable omegas. The reason is twofold and straightforward:
(i) The power set is a σ-field.

(ii) Any set in the power set must be in the σ-field generated, since countable unions have to be, and we can count over all elements in the generating set, while including or excluding (including the empty set) every such element.

Generating $(\mathbf{R},\mathcal{B}(\mathbf{R}))$:

Whereas in the former case the set of all singleton sets with every element was already enough to generate the power set, in the reals this approach does not yield a σ-field that is rich enough to support practical purposes. The reason is that the above method of generating a σ-field results in this case in a field that contains only sets that are either countable or have countable complements. (A look back to the definition of σ-fields and the minimum-property of generated σ-fields conveys conviction why this must be.) As a consequence, the unit interval does not lie in this σ-field! (It contains uncountably many points, and so does the complement.) For this reason, we strive for a larger σ-field. Using the algebra $\mathcal{R}$, containing all finite unions of intervals (we used left-open right-closed ones) on $\mathbf{R} \cup \{-\infty, +\infty\}$, as a generator, we obtain the Borel σ-field. All sets therein are by definition Borel sets. (Note that there are subsets of the reals that are not Borel sets. An example would be the middle thirds Cantor set.)

Generating $(\mathbf{R^d},\mathcal{B}(\mathbf{R^d}))$:

This is completely analogous to the previous case, just that we do not use the algebra of finite unions of intervals as a generator, but the algebra of finite unions of (left-open, right-closed) parallelotops.

Review 4 (Test Feb-05), *1.44 (Notes Feb-01)*

PROBLEM: State the measure extension theorem. Show how to apply this theorem for defining Borel measures on $\mathbb{R}$.

—–

The Measure Extension Theorem:

"Every finite σ-additive content $\mu|\mathcal{A}$ defined on a field has a uniquely determined measure extension to $\mathcal{F} = \sigma(\mathcal{A})$"

We use this theorem primarily in the following way: Defining a measure for Borel sets directly would constitute a considerable difficulty. However, by construction we know that the Borel σ-field is generated by the field $\mathcal{R}$ of finite unions of intervals (see problem 1.8 script version 5th of Feb p. 4 why $\mathcal{R}$ is a field, and review question 3 on how to generate a σ-field from a field or an arbitrary set of sets). For such intervals it is easy to specify an intuitive content: The length of an interval is its content, and since any finite union of intervals can unambiguously be decomposed into the union of finitely many pairwise disjoint intervals, also the respective content can be defined in a straightforward way as the sum of the involved intervals' contents (and would hence be uniquely determined). By the measure extension theorem this σ-additive content on $\mathcal{R}$ (which is not a σ-field) can be extended to a measure on $\sigma(\mathcal{R})$. This is the Borel measure.
NOTE: This works again completely analogously in $\mathbf{R}^d$, with the only difference that we define the content of a figure in $\mathbf{R}^d$ to be the product of its lengths in every dimension.

Review 5 (Test Feb-05), *2.10 (Notes Feb-01)*

> PROBLEM: Explain the abstract concept of a measurable function. State the basic abstract properties of measurable functions.

Let$(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $(Y, \mathcal{B})$ be a measurable space and the function $f : \Omega \mapsto Y$

Definition: A function $f : (\Omega, \mathcal{A}, \mu) \mapsto (Y, \mathcal{B})$ is called $(\mathcal{A}, \mathcal{B})$-measurable if $f^{-1}(B) \in \mathcal{A} \; \forall \; B \in \mathcal{B}$.

If $f : (\Omega, \mathcal{A}, \mu) \mapsto (Y, \mathcal{B})is(\mathcal{A}, \mathcal{B})$-measurable, then we may define:

$$\mu^f(B) := \mu(f \in B) = \mu(f^{-1}(B))$$

with $B \in \mathcal{B}$ This is the image of μ under f or the distribution of f under μ.

Intuitive Backround using random variables:

A random variable is a measurable real-valued function on Ω. The arbitrary r.v. X is a function from a probability space $(\Omega, \mathcal{A}, P)$ to $\Re$. We are usually interested in probabilities $P(X \in B)$ which is the distribution

$$P^X(B) = P(X \in B) \; \forall B \in \mathcal{B}$$

For defining the distribution function it is important that $P(X \in B)$ makes sense. This is true iff the inverse image $(X \in B) = X^{-1}(B)$ is in $\mathcal{A}$. Therefore, $X : \Omega \mapsto \Re$ cannot be a arbitrary function but must satisfy the measurability property:

$$(X \in B) \in \mathcal{A} \; \forall B \in \mathcal{B}$$

Basis Abstract Properties:

Measurability property: $(X \in B) \in \mathcal{A} \; \forall B \in \mathcal{B}$.

Let $f : (\Omega, \mathcal{A}) \mapsto (Y, \mathcal{B})$ be $(\mathcal{A}, \mathcal{B})$ -measurable, let $g : (Y, \mathcal{B}) \mapsto (Z, \mathcal{C})$ be $(\mathcal{B}, \mathcal{C})$ -measurable, then $f \circ g$ is $(\mathcal{A}, \mathcal{C})$-measurable.

Let $f : (\Omega, \mathcal{A}) \mapsto (Y, \mathcal{B})$ and let $\mathcal{C}$ be a generating system of $\mathcal{B}$ i.e. $(\mathcal{B} = \sigma(\mathcal{C}))$. Then f is $(\mathcal{A}, \mathcal{B})$-measurable iff $f^{-1}(C) \in \mathcal{A}\ \forall C \in \mathcal{C}$

Review 6 (Test Feb-05), *2.20 (Notes Feb-01)*

PROBLEM: Describe the structure of the set of real-valued measurable functions. Explain the role of simple functions.

Let $(\Omega, \mathcal{F})$ be a measurable space and let $\mathcal{L}(F)$ be the set of all $\mathcal{F}$-measurable functions.

Criterions for checking measurability of real-valued functions (structure):

- $f : \Omega \mapsto \Re$ is $\mathcal{F}$ -measurable iff $(f \leq \alpha) \in \mathcal{F}$ for every $\alpha \in \Re$
- Let $(f_1, f_2, \dots, f_n)$ be measurable functions, then $f = (f_1, f_2, \dots, f_n) : \Omega \mapsto \Re^n$ is $(\mathcal{F}, \mathcal{B}^n)$ -measurable.
- Let$(f_1, f_2, \dots, f_n)$be measurable functions. Then for every continuous function $\Phi : \Re^n \mapsto \Re$ The composition $\Phi(f_1, f_2, \dots, f_n)$is measurable.
- Let$(f_1, f_2, \dots, f_n)$ be measurable functions. Then $f_1 + f_2, f_1 * f_2, f_1 \cap f_2, f_1 \cup f_2$ are measurable functions.

$\mathcal{L}(F)$ is a space of functions where we can perform any algebraic operation without leaving the space $\Rightarrow$ can even do it when involving a countable set of measurable functions: Let $(f_n)_{n \in \aleph}$ be a sequence of measurable functions. Then $\sup_n f_n$ and $\inf_n f_n$ are measurable functions. Let $A := (\exists \lim_n f_n)$. Then $A \in \mathcal{F}$ and $\lim_n f_n * 1_A$ is measurable. Which means that any function that can be written as an expression of countable many operations with countable many measurable functions is measurable.

How do typical measurable functions look like? Role of simple functions?

- Every measurable function f is the limit of some sequence of simple measurable functions!
- If f is bounded then the approximating sequence can be chosen to be uniformly convergent
- If $\leq$ then the approximating sequence can be chosen to be increasing: Let $f \leq$ and define:

$$f_n := \begin{cases} \frac{k-1}{2^n} \text{ if } \frac{k-1}{2^n} \leq f < \frac{k}{2^n} n \\ n \text{ if } f \geq n \end{cases}$$

Review 7 (Test Feb-05), *2.27 (Notes Feb-01)*

PROBLEM: Explain how the measure extension theorem is applied to construct probability spaces and random variables with given distributions.

———

Let (Ω_t, F_t, P_t) , t$\in$ T be probability spaces.
That is: $P_t(\Omega_t) = 1 \; \forall t$
$P_t(\phi) = 0 \; \forall t$

Let l_t be the class of all measurable cylinders of the form:
$\prod_{t \in T_N} A_t \times \prod_{t \in T - T_N} \Omega_t$ in the T-dimensional product measurable space $(\prod \Omega_t \; , \prod F_t)$.

The Class β_T of all finite sums of these cylinders is a field and the minimal σ-field F_T over β_T is the product σ-field $\prod F_t$ by definition. The product probability $P_T = \prod P_t$ on l_t is defined by assigning the product of probabilities of its sides to every interval cylinder that is:

$$P_T(\prod_{t \in T_N} A_t \times \prod_{t \in T - T_N} \Omega_t) = \prod_{t \in T_N} P_t(A_t). \prod_{t \in T - T_N} P_t(\Omega_t) = \prod_{t \in T_N} P_t(A_t)$$

Then $P_t(\Omega_T) = 1$ and P_T on l_T is finitely additive and determines its extension to a finitely additive set function P_T on β_T

Review 8 (Test Feb-05), *3.30 (Notes Feb-01)*

PROBLEM: Describe the process of constructing the integral, beginning with indicators and ending with integrable functions.

- Let $f = 1_F$ be an $\mathcal{F}$-measurable indicator function. We define the μ-integral of f as follows: $\int f\, d\mu := \mu(F)$.
- Let $f = \sum_{i=1}^n a_i 1_{F_i}$ be a nonnegative simple $\mathcal{F}$-measurable function ($f \in \mathcal{S}(\mathcal{F})$) with its canonical representation. Then $\int f\, d\mu := \sum_{i=1}^n a_i\, \mu(F_i)$.
- Let f be a nonnegative measurable function ($f \in \mathcal{L}^+(\mathcal{F})$) and $f_n \uparrow f$, where $f_n \in \mathcal{S}^+(\mathcal{F})$. The μ-integral is defined as $\int f\, d\mu := \lim_n \int f_n\, d\mu$.
- Let $f \in \mathcal{L}^1(\mu)$, i.e. $\int f^+\, d\mu < \infty$ and $\int f^-\, d\mu < \infty$. Then we define $\int f\, d\mu := \int f^+\, d\mu - \int f^-\, d\mu$, where f^+, $f^- \in \mathcal{L}^+(\mathcal{F})$.

Review 10 (Test Feb-05), *6.12 (Notes Feb-01)*

PROBLEM: Explain the notion and the definition of conditional expectations. State three properties which you consider as most important for this concept.

Let (Ω,F,P) be a probability space and let $\mathcal{A} \subseteq \mathcal{F}$ be a sub σ-field. If a random var x is $\mathcal{A}$ measurable, then the information in $\mathcal{A}$ tells us $\forall$ about x.

However if x is not $\mathcal{A}$ measurable then we may be interested in finding the best $\mathcal{A}$ measurable approximation of x. This leads to the concept of conditional expectation. Consider x=y+R where y is $\mathcal{A}$ measurable and R is uncorrelated with $\mathcal{A}$.
If we require E(y)=E(x) $\Rightarrow$ E(R)=0 $\Rightarrow$ $\int_A RdP$=0 $\forall A \in \mathcal{A}$

$\int_A xdP$= $\int ydP$ for all A$\in$ $\mathcal{A}$

For this integrals to be defined we need nonnegative and integrable random variables

Definition: Let $(\Omega, \mathcal{F}, D)$be a probability space and let $\mathcal{A} \subseteq \mathcal{F}$ be a sub σ -field. Let x be a nonnegative or integrable random var. Then conditional expectation $E(x|\mathcal{A})$ of x is nonnegative (resp integrable) $\mathcal{A}$ measurable random variable y satisfying:

$\int_A xdP$=$\int_A ydP$ forall A $\in \mathcal{A}$ and $y = E(x|\mathcal{A})$

Basic properties

If x is an integrable random variable then:

1. $E(E(x|\mathcal{A})$=E(x)

2. if x is $\mathcal{A}$ measurable then $E(x|\mathcal{A})$=x

3. If x is independent of $E(x|\mathcal{A})$ then $E(x|\mathcal{A})$=E(x)

4. Linearity: if x and y are nonnegative random variables then $E(\alpha x + \beta y|\mathcal{A}) = \alpha E(x|\mathcal{A}) + \beta E(y|\mathcal{A})$

Review 11 (Test Feb-05), *7.4 (Notes Feb-01)*

> PROBLEM: Explain the ruin problem and derive intuitively the difference equations for ruin probabilities.

——

Ruin Problem: A person decides to try to increase the amount of money in his/her pocket by participating in some gambling. Initially, the gambler in question has a certain amount of money, say a. The gambler decides that he/she will gamble until a certain goal, c, is achieved or there is no money left. If the gambler achieves the goal of c he/she will stop playing. If the gambler ends up with no money he/she is ruined (thus the name of the problem). So, the gambler starts playing a game of chance (e.g., poker, roulette, slot machines, etc.). The question is: What chance does the gambler have of achieving the goal?

Analytically this problem can be written as:

$$X_i \begin{cases} +1 & \text{with probability p} \\ -1 & \text{with probability (1-p)} \end{cases}$$

Let's denote the sum of the random variables + "a" (the initial capital) as V_0 , so we get:

$$V_0 = a + X_1 + X_2 + X_3 + \ldots + X_n$$

with the $X_i's$ as independent random variables. This sequence of r.v. $(V_n)_{n\geq 0}$ is called a random walk. The question is when the random walk hits the boundary "a+c"? Let's define $T_x = min(n : V_n = x)$ If T_0 ¡ T_c then the gambler has lost all his wealth before hitting the boundary and faces the ruin situation. Hence the probabilities of ruin and winning will be:

$$q_0(a) = p(T_0 < T_c|V_0 = a)$$
$$q_c(a) = p(T_c < T_0|V_0 = a)$$

with q_0 and q_c depending on a sequence of random variables. So we can write

$$p(T_c < T_0|V_0 = a) = p * (T_c < T_0|V_1 = a+1) + (1-p) * (T_c < T_0|V_1 = a-1)$$

Or equivalently

$$q_c(a) = q_c(a+1) + q_c(a-1) \text{for} 0 < a < c \text{and} \quad q_c(0) = 0, q_c(c) = 1$$

The difference equation comes from the fact that the random walk hast the same ruin probability no matter at which point in time we start from. It was obtained by splitting up events which can be done more generally whenever

$$P(A|B) = P(A|C_1) * P(C_1|B) + P(A|C_2) * P(C_2|B) \text{ if } \quad B = C_1 \cup C_2$$

In our case $a = (a+1) \cup (a-1)$

Review 12 (Test Feb-05), *7.22 (Notes Feb-01)*

PROBLEM: Explain the notions of filtration and stopping time for stochastic sequences. Show the importance of Wald's equation by a typical application.

——

The pre-requisite for a (financial) stochastic sequence to be realised are investors who are willing to engage in a certain gamble. Comparisons between different gambles and decisions for or against gambles are based upon the information available on the gambles at the time of evaluation. In the language of stochastic sequences, this information is called the *past* of a sequence (X_i), i.e. the σ-field $\mathcal{F}_k := \sigma(X_1, X_2, ...X_k)$

generated by the events $(X_1 \in B_1, X_2 \in B_2, ...X_k \in B_k)$ (see Def. 7.7. (notes 2006-02-01)).

If the past of a certain stochastic sequence is as such that the investor can subjectively or objectively deduct from it a pontential for a future development, which is in line with his investment strategy, he will eventually engage in the respective gamble, at the point in time following the evaluation at time k. In this case, $(\sigma = k) \in \mathcal{F}_k$.

Certainly, if the starting time $(\sigma = k) \in \mathcal{F}_k$, this implies that the *stopping time* $(\tau = k) \notin \mathcal{F}_k$, as otherwise it would not be sensible to engage in the gamble. Then it must be the case that $(\tau = m) \in \mathcal{F}_m$ for some $m > k$. The formal notion of a stopping time is provided by Definition 8.24 (notes 2006-02-01): "A random variable $\tau : \Omega \rightarrow [0, \infty]$ is called a *stopping time* if $(\tau \leq t) \in \mathcal{F}_t \forall t \geq 0$". This means that the information sets entailed in the sequence of σ-fields $(\mathcal{F}_k)_{k \geq 0}$ are increasing, i.e. the sequence is called a *filtration* in the terminology of stochastic sequences (see Def. 7.9. (notes 2006-02-01)). If this were not the case, or at least if the investor could not expect it from the evaluation of the past, this would stand at odds with the realisation of any investment plan, so that the gamble would never be started. Therefore, it can be expected that $(\tau = m) \in \mathcal{F}_m$ for some $k < m < \infty$, i.e. that the stopping time is bounded.

For any decision upon an investment plan (i.e. the engagement in a gamble), one of the core indicators is of course exactly the concrete value of this expectation, $E(\tau)$. For its calculation, one can use the past of a sequence to derive the expected accumulated gain at the stopping time, $E(S_\tau)$, and the expected gain for each X_i, and then apply *Wald's equation* (see Theorem 7.16 (notes 2006-02-01)):

$$E(S_\tau) = \mu E(\tau)$$

where $\mu = E(X_k)$, and therefore

$$E(\tau) = \frac{E(S_\tau)}{\mu}$$

(See also problem E26 for such an application!)

Review 13 (Test Feb-05), *7.30 (Notes Feb-01)*

> PROBLEM: Explain how gambling systems lead in a natural way to the notion of a martingale.

——

Just look at Theorem 7.26 (Optional stopping for gambling systems, Notes 5/2). By additivity of expectation we have $E(V_\tau - V_\sigma) = E(V_\tau) - E(V_\sigma)$. Now use Y_n instead of V_n. Then in case of $\mu = 0$ we have a martingale, in case of $\mu \leq 0$ a supermartingale and a submartingale if $\mu \geq 0$.

Review 14 (Test Feb-05), *8.8 (Notes Feb-01)*

> PROBLEM: Motivate the concept of a Wiener process at hand of random walks. Explain the term "increments independent of the past".

——

Let $X_1, X_2, \ldots, X_n$ be independent and in addition $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$. Then $S_n = X_1 + X_2 + \ldots + X_n$ whenever $n = 1, 2, \ldots$ is a symmetric random walk. As $X_1, X_2, \ldots, X_n$ are independetn, the increments $S_n - S_m = X_{m+1} + \ldots + X_n$ are also independent. Moreover, we have

$$E(X_i) = 0 \Rightarrow E(S_n - S_m) = 0 \text{ and}$$

$$V(X_i) = 1 \Rightarrow V(S_n - S_m) = n - m$$

Thus, the Wiener process can be interpreted as a continuous time version of a symmetric random walk.

The past process $(X_t)_{t \geq 0}$ at time t is the σ-field of events $\mathcal{F}_t^X \sigma(X_s : s \leq t)$ generated by variables X_s of the process prior to t, i.e. $s \leq t$. The intuitive idea behind the concept of past is the following: $\mathcal{F}_t^X$ consists of all the events which are observable if one observes the process up to time t. It represents the information about the process available at time t.

Let $s_1 \leq s_2 \leq \ldots \leq s_n \leq s \leq t$ Then the random variables $W_{s_1}, W_{s_2} - W_{s_1}, \ldots, W_{s_n} - W_{s_{n-1}}, W_t - W_s$ are independent. It follows that even the random variables $W_{s_1}, W_{s_2}, \ldots W_{s_n}$ are independent of $W_t - W_s$. Since this is valid for any choice ot fime points $s_i \leq s$ the independence assertion carries over to the whole past $\mathcal{F}_t^X$.

Review 15 (Test Feb-05), *8.13 (Notes Feb-01)*

PROBLEM: In what sense behave the paths of a Wiener process very irregularly ?

There are two kinds of irregularity:

(a) Right continuity:

$$\lim_{x_n \to x^+} f(x_n) = f(x) \text{ but } \lim_{x_n \to x^-} f(x_n) \neq f(x)$$

(b) Left continuity:

$$\lim_{x_n \to x^-} f(x_n) = f(x) \text{ but } \lim_{x_n \to x^+} f(x_n) \neq f(x)$$

Moreover, the paths of the Wiener process are continuus but almost nowhere differentiable.